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3 (Sem-5/CBCS) MAT HC 1

2021

(Held in 2022)

**MATHEMATICS**

(Honours)

Paper : MAT-HC-5016

**(Riemann Integration and Metric Spaces)**

Full Marks : 80

Time : Three hours

**The figures in the margin indicate full marks for the questions.**

1. Answer the following as directed :  $1 \times 10 = 10$
- (a) Describe an open ball in the discrete metric space.
- (b) Find the derived set of the sets  $(0, 1]$  and  $[0, 1]$ .
- (c) A subset  $B$  of a metric space  $(X, d)$  is open if and only if
- (i)  $B = \bar{B}$
  - (ii)  $B = B^\circ$
  - (iii)  $B \neq \bar{B}$
  - (iv)  $B \neq B^\circ$

(Choose the correct one)

Contd.

(d) Which of the following is false ?

(i)  $\phi^\circ = \phi, X^\circ = X$

(ii)  $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$

(iii)  $(A \cap B)^\circ = A^\circ \cap B^\circ$

(iv)  $(A \cup B)^\circ = A^\circ \cup B^\circ$

where  $A, B$  are subsets of a metric space  $(X, d)$ . (Choose the false one)

(e) The closure of the subset

$F = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$  of the real line  $\mathbb{R}$  is

(i)  $\phi$

(ii)  $F$

(iii)  $F \cup \{0\}$

(iv)  $F - \{0\}$

(Choose the correct one)

(f) In a metric space an arbitrary union of closed sets need not be closed. Justify it with an example.

(g) If  $A$  is a subset of a metric space  $(X, d)$ , then which one is true ?

(i)  $d(A) = d(\overline{A})$

(ii)  $d(A) \neq d(\overline{A})$

(iii)  $d(A) > d(\overline{A})$

(iv)  $d(A) < d(\overline{A})$

(Choose the true one)

(h) When is an improper Riemann integral said to be convergent ?

(i) Evaluate  $\int_0^{\infty} e^{-x} dx$  if it exists

(j) Show that  $\Gamma(1) = 1$

2. Answer the following questions :  $2 \times 5 = 10$

(a) Let  $F$  be a subset of a metric space  $(X, d)$ . Prove that the set of limit points of  $F$  is a closed subset of  $(X, d)$ .

(b) If  $F_1$  and  $F_2$  are two subsets of a metric space  $(X, d)$ , then

$\overline{F_1 \cap F_2} = \overline{F_1} \cap \overline{F_2}$ . Justify whether it is false or true.

(c) Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and  $f: X \rightarrow Y$ . If for all subsets  $A$  of  $X$ ,  $f(\overline{A}) \subseteq \overline{f(A)}$ , then show that  $f$  is continuous on  $X$ .

(d) Let  $f: [a, b] \rightarrow \mathbb{R}$  be integrable. Show that  $|f|$  is integrable.

(e) Show that the function  $f: [a, b] \rightarrow \mathbb{R}$  defined by  $f(x) = c$  for all  $x \in [a, b]$  is integrable with its integral  $c(b-a)$ .

3. Answer **any four** parts : 5×4=20

(a) Define a complete metric space. Show that the metric space  $X = \mathbb{R}^n$  with the metric given by

$$d_p(x, y) = \left( \sum |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1$$

where  $x = (x_1, x_2, \dots, x_n)$  and

$y = (y_1, y_2, \dots, y_n)$  are in  $\mathbb{R}^n$ , is a complete metric space. 1+4=5

(b) Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Prove that a mapping  $f: X \rightarrow Y$  is continuous on  $X$  if and only if  $f^{-1}(G)$  is open in  $X$  for all open subsets  $G$  of  $Y$ . 5

(c) Prove that if the metric space  $(X, d)$  is disconnected, then there exists a continuous mapping of  $(X, d)$  onto the discrete two-element space  $(X_0, d_0)$ . 5

(d) Let  $f: [a, b] \rightarrow \mathbb{R}$  be a continuous function. Prove that  $f$  is integrable. 5

(e) Discuss the convergence of the integral  $\int_1^{\infty} \frac{1}{x^p} dx$  for various values of  $p$ . 5

(f) Show that for  $a > -1$ ,

$$S_n = \frac{1^n + 2^n + \dots + n^n}{n^{1+a}} \rightarrow \frac{1}{1+a}. \quad 5$$

4. Answer **any four** parts :  $10 \times 4 = 40$

(a) (i) Let  $(X, d)$  be a metric space.  
Define  $d : X \times X \rightarrow \mathbb{R}$  by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} \text{ for all}$$

$x, y \in X$ . Prove that  $d'$  is a metric on  $X$ .

Also show that  $d$  and  $d'$  are equivalent metrics on  $X$ .

4+2=6

(ii) Prove that a convergent sequence in a metric space is a Cauchy sequence. 4

(b) (i) Let  $(X, d)$  be a metric space and  $F$  be a subset of  $X$ . Prove that  $F$  is closed in  $X$  if and only if  $F^c$  is open. 5

(ii) If  $(Y, d_Y)$  is a subspace of a metric space  $(X, d)$ , then show that a subset  $Z$  of  $Y$  is open in  $Y$  if and only if there exists an open set  $G \subseteq X$  such that  $Z = G \cap Y$ .

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(c) Prove that a metric space  $(X, d)$  is complete if and only if for every nested sequence  $\{F_n\}_{n \geq 1}$  of non-empty closed subsets of  $X$  such that  $d(F_n) \rightarrow 0$  as  $n \rightarrow \infty$ , the intersection  $\bigcap_{n=1}^{\infty} F_n$  contains one and only one point. 10

(d) (i) Prove that in a metric space  $(X, d)$ , each open ball is an open set. 4

(ii) Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and  $A \subseteq X$ . Prove that a function  $f : A \rightarrow Y$  is continuous at  $a \in A$  if and only if whenever a sequence  $\{x_n\}$  in  $A$  converges to  $a$ , the sequence  $\{f(x_n)\}$  converges to  $f(a)$ . 6

(e) (i) Define uniformly continuous mapping in a metric space. Give an example to show that a continuous mapping need not be uniformly continuous. 1+4=5

(ii) Prove that the image of a Cauchy sequence under a uniformly continuous mapping is itself a Cauchy sequence. 5

(f) Let  $(\mathbb{R}, d)$  be the space of real numbers with the usual metric. Prove that a subset  $I \subseteq \mathbb{R}$  is connected if and only if  $I$  is an interval. 10

(g) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Show that  $f$  is integrable if and only if it is Riemann integrable. 10

(h) (i) State and prove first fundamental theorem of calculus. Using it show that

$$\int_0^a f(x) dx = \frac{a^4}{4} \text{ for } f(x) = x^3.$$

1+3+2=6

(ii) Let  $f$  be continuous on  $[a, b]$ . Prove that there exists  $c \in [a, b]$

$$\text{such that } \frac{1}{b-a} \int_a^b f(x) dx = f(c).$$

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