Total number of printed pages–16

3 (Sem–6/CBCS) MAT HC 1 (N/O)

2025

MATHEMATICS

(Honours Core)

Paper : MAT–HC–6016

[New Syllabus]

(Riemann Integration and Metric Spaces)

Full Marks : 80

Time : Three hours

[Old Syllabus]

(Complex Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

[New Syllabus]

(Riemann Integration and Metric Spaces)

Full Marks : 80

Time : Three hours

1. Answer the following as directed : $1\times10=10$

(a) A bounded function $f:[a,b]\to\mathbb{R}$ is integrable if for each $\varepsilon>0$, there exists a partition P such that

(i) $U(f,P)<\varepsilon+L(f,P)$

(ii) $U(f,P)<\varepsilon-L(f,P)$

(iii) $U(f,P)>\varepsilon+L(f,P)$

(iv) $U(f,P)>\varepsilon-L(f,P)$

(Choose the correct option)

(b) State mean value theorem for integrals.

(c) Evaluate $\Gamma\frac{3}{2}$.

(d) Define Euclidean metric on $\mathbb{R}^n$.

(e) The open ball $S\left(\frac{1}{2},1\right)$ on the usual metric space $(\mathbb{R},d)$ is

(i) $\left(\frac{1}{2},\frac{3}{2}\right)$

(ii) $\left(\frac{1}{2},-\frac{3}{2}\right)$

(iii) $\left(-\frac{1}{2},\frac{3}{2}\right)$

(iv) $\left(-\frac{1}{2},-\frac{3}{2}\right)$

(Choose the correct option)

(f) Let X be a non-empty set. If $d:X\times X\to\mathbb{R}$ is a pseudometric on X, then which of the following statement is false ?

(i) $d(x,y)\geq0$ for all $x,y\in X$

(ii) $d(x,y)=0\Rightarrow x=y$ for all $x,y\in X$

(iii) $d(x,y)=d(y,x)$ for all $x,y\in X$

(iv) $d(x,y)\leq d(x,z)+d(z,y)$ for all $x,y,z\in X$

(Choose the correct option)

(g) If A is a non-empty subset of a metric space (X, d) such that A^c is closed in X, then A is

(i) closed in X

(ii) open in X

(iii) Both open and closed in X

(iv) None of the above

(Choose the correct option)

(h) Show that the closure $\overline{F}$ of $F \subseteq X$, where (X, d) is a metric space, is closed.

(i) Define a contraction mapping on a metric space.

(j) Which of the following statements are true ?

(i) A singleton set $\{x\}$ in any metric space is always connected.

(ii) The interval $[2, 3)$ is not connected in the usual metric space $(\mathbb{R}, d)$.

(iii) If (X, d) is a connected metric space, there exists a proper subset of X which is both open and closed in X.

(iv) Closure of a connected set in a metric space is connected.

(Choose the correct option)

2. Answer the following questions : 2×5=10

(a) Let $f(x) = x$ on $[0, 1]$ and

$$P = \left\{x_i = \frac{i}{8}, i = 0,1,2,....8\right\}$$

Find $L(f, P)$ and $U(f, P)$

(b) Prove that $\overline{|(\alpha+1)} = \alpha\overline{|\alpha}$

(c) Show that the discrete metric space is a complete metric space.

(d) Let (X, d) be a metric space and $\overline{S}(x, r) = \{y \in X : d(x, y) \le r\}$ be a closed ball in X. Prove that $\overline{S}(x, r)$ is closed.

(e) Prove that if Y is a connected set in a metric space (X, d), then any set Z such that $Y \subseteq Z \subseteq \overline{Y}$ is connected.

3. Answer ***any four*** questions : 5×4=20

(a) Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that f is integrable.

(b) Show that $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2n+k} = \log\frac{3}{2}$

(c) Define an open ball in a metric space. Prove that in any metric space (X, d), each open ball is an open set. 1+4=5

(d) Let (X, d_X) and (Y, d_Y) be two metric spaces. Prove that a mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y.

(e) A continuous function may not map a Cauchy sequence into a Cauchy sequence — Justify it.

Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$ be uniformly continuous. If $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in X, then show that $\{f(x_n)\}_{n\geq 1}$ is also a Cauchy sequence in Y. 1+4=5

(f) Let (X, d_X) be a metric space. If every continuous function $f : (X, d_X) \to (\mathbb{R}, d)$ has the intermediate value property, then prove that (X, d_X) is a connected metric space.

Answer ***either*** *(a)* ***or*** *(b)* of the following questions : (Q.4 to Q.7) 10×4=40

4. (a) (i) State and prove First Fundamental Theorem of Calculus. 1+4=5

(ii) Discuss the convergence of the integral $\int_1^\infty \frac{1}{x^p} dx$ for various values of p. 5

(b) (i) Show that $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x^n$ is integrable and $\int_0^1 f(x)dx = \frac{1}{n+1}$. 4

(ii) Let f be continuous on $[a, b]$. Prove that there exists $c \in [a, b]$ such that $\frac{1}{b-a}\int_a^b f(x)dx = f(c)$.

Use the 1st mean value theorem to prove that for $0 < a \leq 1$ and $n \in \mathbb{N}$, $\int_0^1 \frac{x^n}{1+x} dx \to 0$ as $n \to \infty$.

3+3=6

5. (a) (i) Let $X = \mathbb{R}$. For $x, y \in \mathbb{R}$, define $d(x, y) = \frac{|x-y|}{1+|x-y|}$. Show that d is a metric on $\mathbb{R}$. 4

(ii) Prove that a convergent sequence in a metric space is a Cauchy sequence.

Does the converse of this hold? Justify it. 4+2=6

(b) (i) Prove that the metric space $X = \mathbb{R}^n$ with the metric given by

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1$$

where $x = (x_1, x_2, \ldots\ldots, x_n)$ and $y = (y_1, y_2, \ldots\ldots, y_n)$ are in $\mathbb{R}^n$, is a complete metric space. 5

(ii) Let (X, d) be a metric space and F_1, F_2 be subsets of X. Prove that $(F_1 \cup F_2)' = F_1' \cup F_2'$ and $\overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2}$. 3+2=5

6. (a) (i) Let (X, d) be a metric space and let $x \in X$ and $A \subseteq X$ be non-empty. Then prove that $x \in \overline{A}$ if and only if $d(x, A) = 0$. 4

(ii) Let (X, d_X) and (Y, d_Y) be metric spaces and $A \subseteq X$. Prove that a function $f : A \to Y$ is continuous at $a \in A$ if and only if whenever a sequence $\{x_n\}$ in A converges to a, the sequence $\{f(x_n)\}$ converges to $f(a)$. 6

(b) (i) Prove that a mapping $f : X \to Y$ is continuous on X iff $f^{-1}(F)$ is closed in X for all closed subsets F of Y. 4

(ii) Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. Prove that the following statements are equivalent :

I. f is continuous on X

II. $\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$ for all $B \subseteq Y$

III. $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$ 6

7. *(a)* Let $(\mathbb{R}, d)$ be the space of real numbers with the usual metric. Prove that a subset $I \subseteq \mathbb{R}$ is connected if and only if I is an interval. 10

(b) *(i)* If f and g are two uniformly continuous mappings of metric spaces (X, d_X) to (Y, d_Y) and (Y, d_Y) to (Z, d_Z) respectively, then prove that $g \circ f$ is uniformly continuous mapping of (X, d_X) to (Z, d_Z).

Show that the function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous. 4+2=6

(ii) Let (X, d) be a metric space and let $\{Y_\lambda : \lambda \in \wedge\}$ be a family of connected sets in (X, d) having a nonempty intersection. Prove that $Y = \bigcup_{\lambda \in \wedge} Y_\lambda$ is connected. 4

[Old Syllabus]

(Complex Analysis)

Full Marks : 60

Time : Three hours

1. Answer the following questions : $1\times7=7$

   (a) Write down the Cauchy-Riemann equations.

   (b) Define analytic function.

   (c) Find the argument of $\frac{1-i}{1+i}$.

   (d) If $z_1 = 2+i$ and $z_2 = 3-2i$, then evaluate $|3z_1 - 4z_2|$.

   (e) Find $\lim_{z\to i}(z^2+2z)$.

   (f) Find $((3-i)^2-3)i$.

   (g) Express $e^{-i\frac{\pi}{4}}$ in the form $a+bi$.

2. Answer the following questions : $2\times4=8$

   (i) Write $\frac{1-i}{3}$ in the form $re^{i\theta}$.

   (ii) Find $\left|\frac{1+2i}{-2-i}\right|$.

   (iii) Determine the points at which the function $\frac{1}{z-2+3i}$ is not analytic.

   (iv) For any two complex numbers z_1 and z_2, prove that $|z_1z_2| = |z_1||z_2|$.

3. Answer ***any three*** questions : $5\times3=15$

   (a) Prove that $f(z) = z^2 - 2z + 5$ is continuous everywhere in the finite plane.

   (b) Show that $f(z) = e^z$ is analytic at every point of the complex plane.

   (c) Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}$, where C is the circle $|z| = 1$.

   (d) If $f(z) = z^3 - 2z$; $z \in \mathbb{C}$, then find $f'(z)$ at $z = -1$, provided the value exists.

(e) Let $f(z)=\begin{cases} z^2, & z \neq i \\ 0, & z = i \end{cases}$, prove that $f(z)$ is not continuous at $z=i$.

4. Answer ***any three*** questions : 10×3=30

(i) Prove that the necessary and sufficient conditions for the complex function $\omega = f(z) = u(x,y) + iv(x,y)$ to be analytic in a region R are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where all partial derivatives are assumed to be continuous on R.

(ii) If $f(z)$ is analytic with its derivative $f'(z)$ continuous at all points inside and on a simple closed curve C, prove that $\int_C f(z)\,dz = 0$.

(iii) Prove that if $f(z)$ is integrable along a curve C having finite length L and if there exists a positive number M such that $|f(z)| \leq M$ on C, then

$$\left|\int_C f(z)\,dz\right| \leq ML.$$

(iv) (a) Find the analytic function whose real part is
$u = e^{-x}\left[\left(x^2 - y^2\right)\cos y + 2xy \sin y\right]$. 5

(b) Show that the function
$f(z) = \sin x \cosh y + i \cos x \sinh y$
is entire. 5

(v) (a) State and prove Cauchy's Integral Formulae. 7

(b) Evaluate $\frac{1}{2\pi i}\int_C \frac{e^z}{z-2}\,dz$, where C is the circle $|z| = 3$. 3

Contd.

(vi) (a) Suppose that

$z_n = x_n + iy_n, (n = 1,2,3...)$ and $z = x + iy$. Prove that $\lim_{n\to\infty} z_n = z$ if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. 5

(b) Show that, $z^2 e^{3z} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n$, $(|z| < \infty)$. 5
