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Quantum Fisher Information and Entanglement of Moving Two Two-Level Atoms under the Influence of Environmental Effects

Syed Jamal Anwar, M. Usman, M. Ramzan and M. Khalid Khan *

Department of Physics, Quaid-i-Azam University, 45320 Islamabad, Pakistan; sjamalanwar@gmail.com (S.J.A.); usmanqau@hotmail.com (M.U.); mramzan@qau.edu.pk (M.R.)
* Correspondence: mkk@qau.edu.pk

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Abstract: We have investigated numerically the dynamics of quantum Fisher information (QFI) and quantum entanglement (QE) of a two moving two-level atomic systems interacting with a coherent and thermal field in the presence of intrinsic decoherence (ID) and Kerr (non-linear medium) and Stark effects. The state of the entire system interacting with coherent and thermal fields is evaluated numerically under the influence of ID and Kerr (nonlinear) and Stark effects. QFI and von Neumann entropy (VNE) decrease in the presence of ID when the atomic motion is neglected. QFI and QE show an opposite response during its time evolution in the presence of a thermal environment. QFI is found to be more susceptible to ID as compared to QE in the presence of a thermal environment. The decay of QE is further damped at greater time-scales, which confirms the fact that ID heavily influences the system's dynamics in a thermal environment. However, a periodic behavior of entanglement is observed due to atomic motion, which becomes modest under environmental effects. It is found that a non-linear Kerr medium has a prominent effect on the VNE but not on the QFI. Furthermore, it has been observed that QFI and QE decay soon under the influence of the Stark effect in the absence of atomic motion. The periodic response of QFI and VNE is observed for both the non-linear Kerr medium and the Stark effect in the presence of atomic motion. It is observed that the Stark, Kerr, ID, and thermal environment have significant effects during the time evolution of the quantum system.

Keywords: quantum Fisher information (QFI); intrinsic decoherence (ID); thermal environment; two two-level atomic system; Kerr and Stark effect

1. Introduction

Quantum entanglement (QE) is a very mysterious and basic phenomenon of quantum mechanics that describes the correlations between two or more quantum systems [1]. QE most of the time is not only a puzzle but also a resource for quantum information processing (QIP) such as quantum teleportation [2] and quantum metrology [3]. These quantum-information processes depend on entangled states and it is not possible to carry them out with classical resources. This identification has been applied to intensive research for mathematical calculations that would allow a proper quantification. QE can be measured by standard methods such as von Neumann entropy (VNE) [4,5] and Fisher information (FI) [6–8]. The QE of a quantum state can be checked by these measurement tools. Quantum Fisher information (QFI) has a wide application in quantum metrology, which calculates the accuracy of parameter estimation with respect to the quantum Cramer–Rao inequality [9,10]. Recently, QFI has been extensively studied in different fields, which include the study of uncertainty relations [11,12], the calculation of quantum speedup limit time [13], the properties of quantum phase transition [14], and the measurement of entanglement [43].

Entanglement's sudden death" (ESD) and entanglement sudden birth" (ESB) are observed as a result of entanglement measure in some fascinating and striking physical phenomena [16–20]. The QE is generated in the un-entangled qubits after a finite evolution time in the case of sudden birth. The coherent field is an electromagnetic field (EM) that is considered to be more classical than its quantum field [21]. Squeezed, coherent (even and odd) states of the EM field do not have minimum uncertainty, and they are non-classical quantum states and have a large number of applications for quantum communications and in the detection of weak signals [22].

Recently, properties of the Tavis–Cummings model (TCM) were investigated when the time-dependent interaction with field was observed. Open and closed quantum systems were studied in the case of QE between two atoms (qubits) [23,24]. Moreover, a three-photon process was discussed in the case of the QE of two moving atoms interacting with a single-mode field [25]. In [26], the authors studied the atomic Wehrl entropy (AWE) of a V-type three-level atomic system interacting with a two-mode squeezed vacuum state, and the results showed that atomic motion and mode structure play significant and prominent roles in the evolution of AWE.

There is always a possibility that the real physical system naturally interacts with the surrounding environment. Decoherence or dissipation may be generated due to the interaction of the quantum system with its surroundings. Recently, Benatti and Floreanini [27] investigated the generation of the QE of two atoms under the effect of a thermal quantum field. The QE response for two accelerated atoms interacted with a bath of fluctuating quantum fields was studied in [28,29]. Quantum correlation and coherence for two atoms coupled with a bath of a fluctuating scalar field were investigated by Huang and Situ [30]. Milburn considers that the system does not evolve continuously; rather, he drives a modified Schrodinger equation that describes the destruction of coherence in the energy basis due to the intrinsic decoherence (ID) [31]. In the model of Milburn, the off-diagonal elements of the density operator are intrinsically suppressed in the energy eigenstate basis because the quantum system evolves under a stochastic sequence of identical unitary transformation; thereby, the ID is investigated. The Milburn model is basically used to investigate decoherence effects in the Jayne–Cummings model (JCM) in open quantum systems, in a two-level system coupled to a photon field and in quantum gates [32–37].

From another point of view, based on the JCM (and of course its generalization), an atom-field entangled state may be produced. For instance, temporal behavior of the VNE of the nonlinear interaction between a three-level atom and a two-mode cavity field in the presence of a cross-Kerr medium and its deformed counterpart [38] and detuning parameters has been investigated in [39,40]. In another case, the amount of the degree of entanglement (DEM) between a three-level atom in motion interacting with a single-mode field in the intensity-dependent coupling regime was calculated [41]. The effects of the mean photon number, the detuning parameter, the Kerr-like medium, and various atom-field couplings on the DEM of the interaction between a three level-atom and a two-mode field were studied in [42]. From another perspective of this field of research, multi-photon JCM may be considered. A multi-photon process is of great importance in atomic systems since it leads to a high degree of correlation between emitted photons, resulting in the non-classical response of emitted light [43,44]. The significance of multi-photon transition becomes more important when the Stark shift is considered. For instance, in two-photon JCM, a single-mode cavity field interacts with a two-level atom through an intermediate state containing emission or absorption [45]. It is very important to note that, when the two atomic levels interact with comparable strength to the intermediate level, the Stark shift becomes prominent and it is not possible to ignore its effect on QE [46,47]. Therefore, by the application of an adiabatic elimination method for the intermediate level(s) of a multi-level atom, such as three- or four-level atoms, the interaction Hamiltonian having a two-level atom along with the Stark shift is determined [48,49]. Under other conditions, a multi-level atom interacting with a single-mode or multi-mode quantized field can be considered as a two-level system under the effect of a Stark shift through the adiabatic elimination method [50–52]. The Stark shift can be usually observed in a two-photon transition [46,47].

In this present paper, our main focus is to investigate the QFI and QE of two two-level moving atomic systems interacting with a coherent field and a thermal field in the presence of ID, a Kerr medium, and a Stark shift. We have calculated the VNE and QFI of the two two-level atomic systems in the presence of atomic motion and without atomic motion. Here, we use QFI, based on a symmetric logarithmic derivative (SLD) operator, to quantify entanglement of two moving two-level atomic systems interacting with a single-mode coherent and a thermal field under the influence of ID. This study mainly focuses on the time evolution of QFI and VNE under the influence ID for moving and non-moving two two-level atomic systems interacting with a coherent and thermal field. The time evolution of the wave function of the complete system is numerically evaluated. Then, the dynamics of the two two-level atoms by considering QFI and VNE are discussed in detail. It is deduced from the numerical results that both the ID and the thermal environment play an important role during the time evolution of the quantum system. QFI and QE show an opposite response during its time evolution in the presence of the thermal environment. The time evolution of QFI is found to be highly sensitive to ID as compared to the QE. Furthermore, the degree of entanglement changes drastically when we increase the ID parameter in the absence of atomic motion. The increased damping behavior of QE validates the basic fact that the system is more prone to ID for greater time-scales in the absence of atomic motion. We study the dynamics of QE and QFI for two two-level atomic systems under the influence of a Stark shift and a non-linear Kerr medium. The time evolution of QFI and entanglement for two three-level atomic systems influenced by the Stark effect and the non-linear Kerr-like medium is investigated. It is found that the Stark effect and the non-linear Kerr medium play dominant roles during the time evolution of the quantum system. The effect of the non-linear Kerr medium is more prominent on the QE as compared to the QFI. Similarly, the Stark effect strongly influences the QE of the two two-level atomic systems. The QFI and QE evolves with time as we increase the Stark effect parameter. Finally, the quantum system is found to be highly sensitive to these environmental influences.

In Sections 2 and 3, we briefly introduce QFI, VNE, and their formulas related to numerical calculations. The Hamiltonian of the system and dynamics are presented in Section 4. In Section 5, we present detailed results and numerical discussions. In Section 6, we present a brief conclusion.

2. Entanglement and Quantum Fisher Information (QFI)

VNE is the most efficient and basic QE measurement tool when the quantum system is in a pure state, so VNE is used to measure entanglement between the field and the two atoms. The VNE is presented in the form of eigenvalues of the atomic density matrix as [5,53]

$$S_{AB} = -\mathrm{Tr}(\rho_{AB} \ln \rho_{AB}) = -\sum_{i=1}^{4} r_i \ln r_i, \tag{1}$$

where r_i are the eigenvalues of the atomic density matrix ρ_{AB}.

The QFI gives the maximum information about the estimated parameter, and the QFI, which is related to θ, can be represented as [54–56]

$$I_\theta = \sum_k \frac{(\partial_\theta \lambda_k)^2}{\lambda_k} + 2 \sum_{k,k'} \frac{(\lambda_k - \lambda_{k'})^2}{(\lambda_k + \lambda_{k'})} |\langle k|\partial_\theta k^{'}\rangle|^2, \tag{2}$$

where $\lambda_k > 0$, $\lambda_k + \lambda_{k'} > 0$, λ_k, and $\lambda_{k'}$ are representing the eigenvalues of the density matrix of state ρ, and k and $k^{'}$ are the corresponding eigenvectors. The first term in Equation (5) is classical Fisher information, and the second term represents its quantum counterpart. In this fashion, we can define the atomic QFI of a bipartite density operator $\rho_{AB.}$ in terms of θ as [57]

$$I_{QF}(t) = I_{AB}(\theta, t) = \text{Tr}[\rho_{AB}(\theta, t) L^2(\theta, t)], \tag{3}$$

where $L(\theta, t)$ is the quantum score [58] (the symmetric logarithmic derivative), which can be found as

$$\frac{\partial \rho_{AB}(\theta, t)}{\partial \theta} = \frac{1}{2}[L(\theta, t)\rho_{AB}(\theta, t) + \rho_{AB}(\theta, t)L(\theta, t)]. \tag{4}$$

3. Intrinsic Decoherence Model

In real practical situations, any quantum system which is said to be open can never be completely isolated from the environment. Milburn [31] gives a simple model of ID by making a modification in standard quantum mechanics. He supposed that, on a sufficiently small time scale τ, the system state takes the form

$$\rho(t + \tau) = \exp[-\frac{i}{\hbar}\theta(\tau)H]\rho(t)\exp[\frac{i}{\hbar}\theta(\tau)H]], \tag{5}$$

with a probability of $\rho(t)$. The master equation presenting the ID under the Markovian approximations is given by

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] - \frac{\gamma}{2}[H, [H, \rho(t)]], \tag{6}$$

where γ is the ID parameter. The above master equation can be expressed by the conventional solution as given below

$$\rho(t) = \sum_{k=0}^{\infty} \frac{(\gamma t)^k}{k!} M^k \rho(0) M^{\dagger k}, \tag{7}$$

where $\rho(0)$ is the density operator of the initial state and M^k is defined as

$$M^k = H^k e^{-iHt} e^{-\frac{\gamma t}{2}H^2}. \tag{8}$$

4. The System Hamiltonian and Its Dynamics

The total Hamiltonian $\hat{H}_T$ under the RWA for a specified system can be described as [59]

$$\hat{H}_T = \hat{H}_{Atom-Field} + \hat{H}_I, \tag{9}$$

where $\hat{H}_{Atom-Field}$ is representing the Hamiltonian for the non-coupling atom and field, and the interaction part is given by $\hat{H}_I$. We will write $\hat{H}_{Atom-Field}$ as

$$\hat{H}_{Atom-Field} = \sum_{k=1}^{2}\sum_{j=0}^{1} \omega_j^{(k)} \hat{\sigma}_{j,j} + \upsilon \hat{a}^\dagger \hat{a}, \tag{10}$$

where $|0\rangle$ is the excited state, and $|1\rangle$ is the ground state of the two-level atom. Index k labels the atom and $\hat{\sigma}_{j,j} = |j\rangle\langle j|$ is the population of the jth state. υ is the frequency of oscillation.

The interaction Hamiltonian of the two two-level atomic systems for the case that is not resonant can be given as [60,61]

$$\hat{H}_I = \sum_{k=1}^{2} \Omega(t) \left[\hat{a} e^{-i\Delta_0^{(k)} t} \hat{\sigma}_{0,1}^{(k)} + \left(\hat{a} e^{-i\Delta_0^{(k)} t} \hat{\sigma}_{0,1}^{(k)}\right)^\dagger\right]. \tag{11}$$

We can define the detuning parameter for the two-level atom as

$$\Delta_0^k = v + \omega_0^{(k)} - \omega_1^{(k)}. \tag{12}$$

In the case of the non-linear Kerr-like medium, the interaction Hamiltonian can be written as

$$\hat{H}_I = \sum_{k=1}^{2} \Omega(t) \left[\hat{a} e^{-i\Delta_0^{(k)} t} \hat{\sigma}_{0,1}^{(k)} + \left(\hat{a} e^{-i\Delta_0^{(k)} t} \hat{\sigma}_{0,1}^{(k)} \right)^{\dagger} \right] + \chi \left(\hat{a}^{\dagger} \hat{a} \right)^2, \tag{13}$$

and when the Stark effect is included in the interaction Hamiltonian, it can be written as

$$\begin{aligned} \hat{H}_I &= \sum_{k=1}^{2} \Omega(t) \left[\hat{a} e^{-i\Delta_0^{(k)} t} \hat{\sigma}_{0,1}^{(k)} + \left(\hat{a} e^{-i\Delta_0^{(k)} t} \hat{\sigma}_{0,1}^{(k)} \right)^{\dagger} \right], \\ &\quad + \beta \hat{a}^{\dagger} \hat{a} \left(|1\rangle \langle 1| \otimes \sigma_0 + \sigma_0 \otimes |1\rangle \langle 1| \right) \end{aligned} \tag{14}$$

where σ_0 is the identity matrix, which is given below

$$\sigma_0 = |0\rangle\langle 0| + |1\rangle\langle |1|,$$

and $\Omega(t)$ represents the shape function of the cavity-field mode and atomic motion is along the [62] z-axis. A realization of particular interest is

$$\begin{aligned} \Omega(t) &= G \ \sin(\eta \pi v t / L) \text{ in the presence of atomic motion, } \eta \neq 0, \\ \Omega(t) &= G \text{ in the absence of atomic motion } \eta = 0, \end{aligned} \tag{15}$$

where the coupling constant for the atom and field is G, v denotes the atomic motion velocity, η is equal to the number of half wavelengths of the mode in the cavity, and L represents the cavity length in the z-direction. The atomic motion velocity is given as $v = \lambda L / \pi$, which leads to

$$\begin{aligned} \Omega_1(t) &= \int_0^t \Omega(\tau) d\tau = \frac{1}{\eta}(1 - \cos(\eta \pi v t / L) \text{ for } \eta \neq 0, \\ &= Gt \text{ for } \eta = 0. \end{aligned} \tag{16}$$

The optimal input state after the phase gate operation for the two two-level atomic systems interacting with a single-mode cavity field can be written as

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|01\rangle + e^{i\phi}|00\rangle) \otimes |\alpha\rangle, \tag{17}$$

where $|1\rangle$ is the ground state, $|0\rangle$ is the excited state of an atom, and α is the coherent state of the input field given as

$$|\alpha\rangle = \sum_{n=0}^{\infty} \alpha^n \sqrt{e^{-|\alpha|^2}/n!} |n\rangle. \tag{18}$$

The optimal input state for the two two-level atomic systems interacting with a thermal field can be written as

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|01\rangle + e^{i\phi}|00\rangle) \otimes \rho_f(0), \tag{19}$$

where $\rho_f(0)$ is the state of input thermal field. We have used thermal state as the input field state in the atom field interaction of the two two-level atomic systems, which is given as

$$\rho_f(0) = \sum_{n=0}^{\infty} P(n)|n\rangle\langle n|\,, \tag{20}$$

where $|n\rangle$ is the fock state, and

$$P(n) = \frac{\bar{n}^n}{(\bar{n}+1)^{(n+1)}}, \tag{21}$$

where $\bar{n}$ is the mean photon number and is given as

$$\bar{n} = (e^{\hbar\omega_f}/k_B T - 1)^{-1},$$

where k_B is the Boltzmann constant, ω_f is the frequency of cavity mode, and T is the temperature.

The wave function $|\Psi(t)\rangle$ in terms of unitary time evolution operator $\hat{U}(t)$ can be written as

$$|\Psi(t)\rangle = \hat{U}\,(t)\,|\Psi(0)\rangle, \tag{22}$$

where $\hat{U}(t)$ is given by

$$U(t) = \sum_{z=1}^{4} \exp(-iE_z t)|\varphi_z(t)\rangle\langle\varphi_z(t)|, \tag{23}$$

where $|\varphi_z(t)\rangle$ and $E_z(t)$ are eigenvectors and eigenvalues of the Hamiltonian H_I, respectively, and γ is the ID parameter. QFI is calculated numerically for the atom-field density matrix given below

$$\hat{\rho}(t) = \hat{U}\,(t)\,\rho(0)\hat{U}^{\dagger}\,(t)\,. \tag{24}$$

One can write the explicit expression of the density matrix as

$$\hat{\rho}(t) = \sum_{m,n}^{4} |\Psi_n(t)\rangle\langle\Psi_n(t)|\hat{\rho}(0)|\Psi_m(t)\rangle\langle\Psi_m(t)|.$$

The density matrix can also be written as [63]

$$\hat{\rho}(t) = \sum_{mn} \exp\left(\frac{\gamma t}{2}\,(E_m - E_n) - i\,(E_m - E_n)\right) \times \langle\Psi_m|\hat{\rho}(0)|\Psi_n\rangle|\Psi_m\rangle\langle\Psi_n|,$$

where $E_{m,n}$ and $\Psi_{m,n}$ are the eigenvalues and corresponding eigenvectors of H_I, respectively. Now, the effect of different environmental parameters γ, θ, and p on the dynamics of QFI and VNE is discussed in detail in the next section.

5. Numerical Results and Discussions

In this section, we will present results of the time evolution of QFI and VNE of a system of the two two-level atomic systems interacting with a coherent field and thermal field under the influence of Stark, Kerr (non-linear), and ID effects. For our convenience, we scaled out the time t, i.e. one unit of time is described by the inverse of the coupling constant G. Initially, we investigate the time evolution of QFI and VNE for two two-level atomic systems interacting with coherent and thermal fields under the influence of ID, a (non-linear) Kerr medium, and a Stark effect with and without atomic motion. In Figures 1 and 2, we plot the QFI and VNE as a function of time for the two two-level atomic systems interacting with a coherent field under the influence of ID for $|\alpha|^2 = 6$, $\gamma = 0$, 0.0001, and 0.001 for phase shift $\phi = 0, \pi/4$ and atomic motion parameter $\eta = 0, 1$. A monotonic relationship between QFI and QE is seen when atomic motion is neglected. It is considered that, as the value of ID parameter is increases, there is decay in both QFI and VNE. This means that QE decreases as the value of ID parameter is increased. The QFI and VNE show periodic response in the presence of atomic motion, and there is no effect of ID on QE in the presence of atomic motion. In Figures 3 and 4, we plot the QFI and VNE as a function of time for the two two-level atomic systems interacting with a thermal field under the influence of ID for $|\alpha|^2 = 6$, $\gamma = 0$, 0.0001, and 0.001 for phase shift $\phi = 0$, $\pi/4$ and atomic motion parameter $\eta = 0, 1$. It is found that the presence of a thermal environment leads to the suppression of QFI during its time evolution without atomic motion in the presence of ID. It is observed that QFI and VNE are highly affected by the thermal field under the influence of an ID, and a decrease is observed in both QFI and VNE. QFI and entanglement exhibit an opposite behavior during the time evolution. A drastic decrease is observed in the behavior of QE due to ID in the presence of a thermal environment. Furthermore, adding the effect of a thermal environment weakens the ID. The damping behavior of entanglement is seen under ID for greater time-scales. The decay of QE is suppressed in the presence of a thermal environment. Therefore, the ID and thermal environment are found to suppress the non-classical effects of the quantum system. However, QE and QFI saturate to a lower level for longer time-scales under environmental influences. Paying attention to the effect of atomic motion parameter η, it is observed that an increase of this parametric quantity and ID results in descending the purity of the state of the system. We find that the QFI and QE show an opposite response during the time evolution in the presence of atomic motion. It is shown that the atomic motion destroys the monotonic correlation between QFI and QE. The periodic behavior of QE in the presence of atomic motion becomes modest under environmental effects.

Finally, according to our numerical results, the decaying of QE becomes faster in a thermal environment as compared to the decay induced by ID. Moreover, these results can be useful to perform long-distance quantum communication, especially when long-living QE is needed and the effect of environments cannot be neglected.

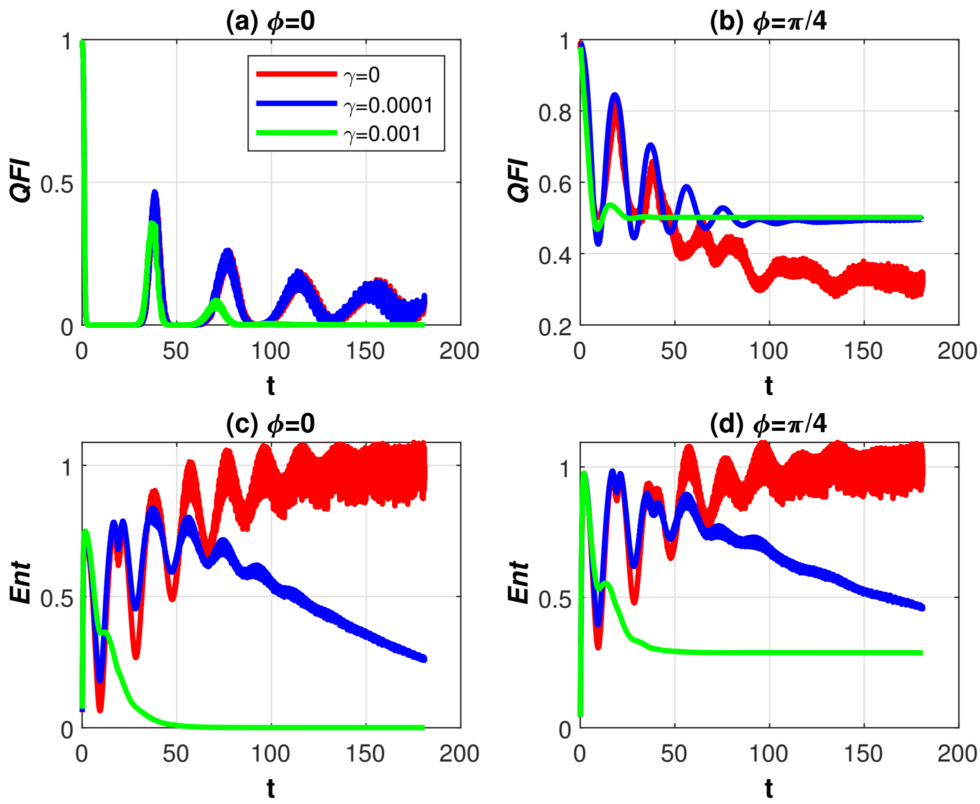

Figure 1. (Color online) The QFI (**upper** panel) and VNE (**lower** panel) as a function of time for two two-level atoms interacting with a coherent field for $\alpha = 6$ in the presence of ID parameter $\gamma = 0, 0.0001, 0.001$ and the phase shift estimator parameters $\phi = 0$ (**left** panel) and $\pi/4$ (**right** panel). The atomic motion parameter η is neglected.

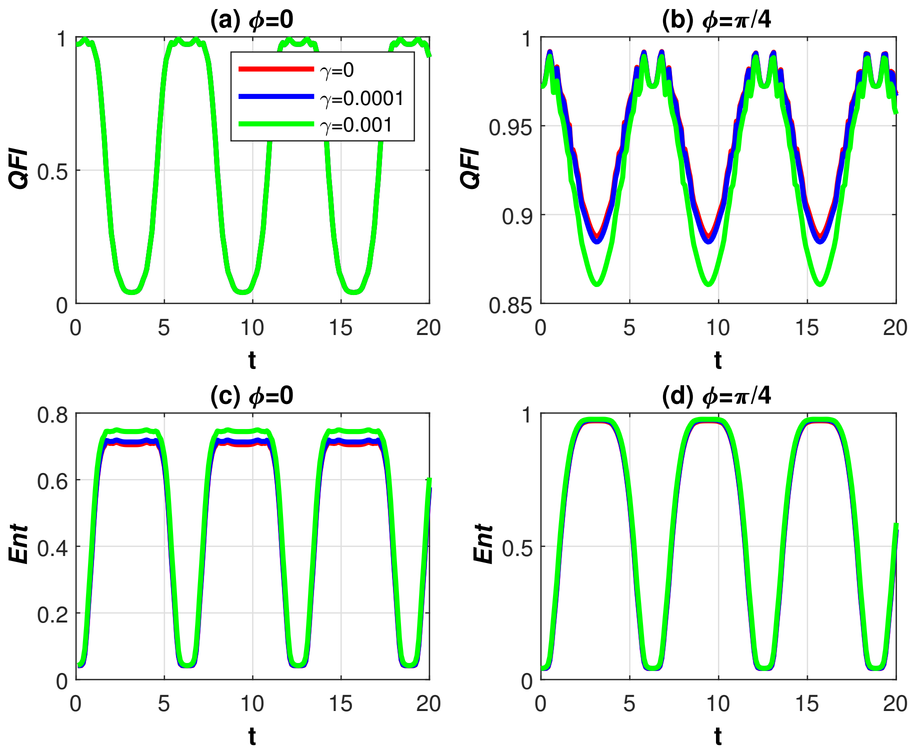

Figure 2. (Color online) The QFI (**upper** panel) and VNE (**lower** panel) as a function of time for two two-level atoms interacting with a coherent field for $\alpha = 6$ in the presence ID parameter $\gamma = 0, 0.0001, 0.001$ and the phase shift estimator parameters $\phi = 0$ (**left** panel) and $\pi/4$ (**right** panel). The atomic motion parameter η is 1.

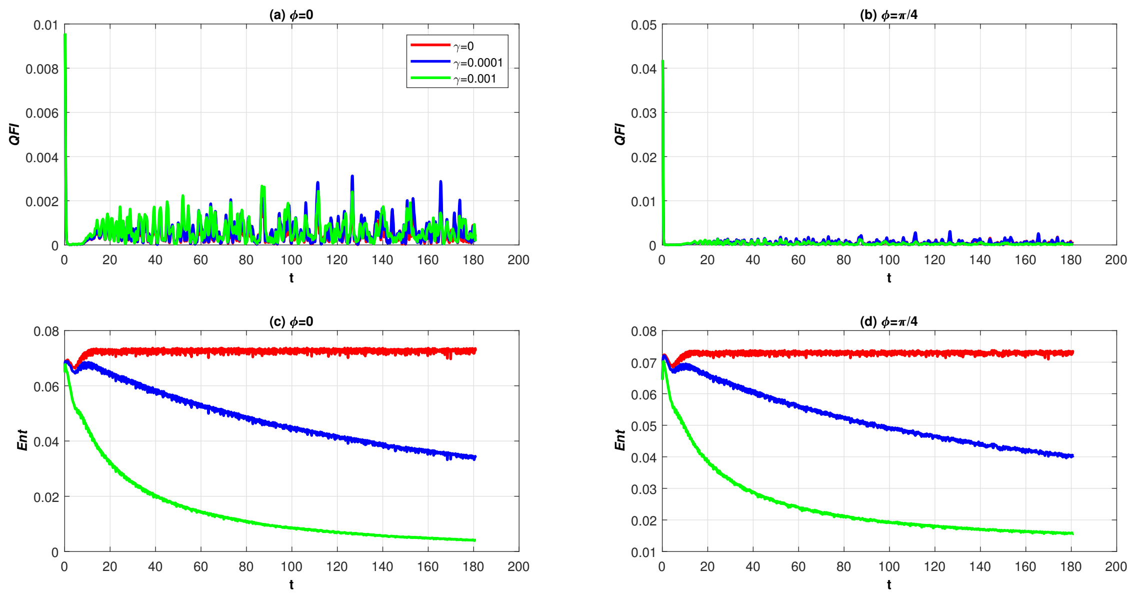

Figure 3. (Color online) The QFI (**upper** panel) and VNE (**lower** panel) as a function of time for two two-level atoms interacting with a thermal field in the presence ID parameter $\gamma = 0, 0.0001, 0.001$ and the phase shift estimator parameters $\phi = 0$ (**left** panel) and $\pi/4$ (**right** panel). The atomic motion parameter η is neglected.

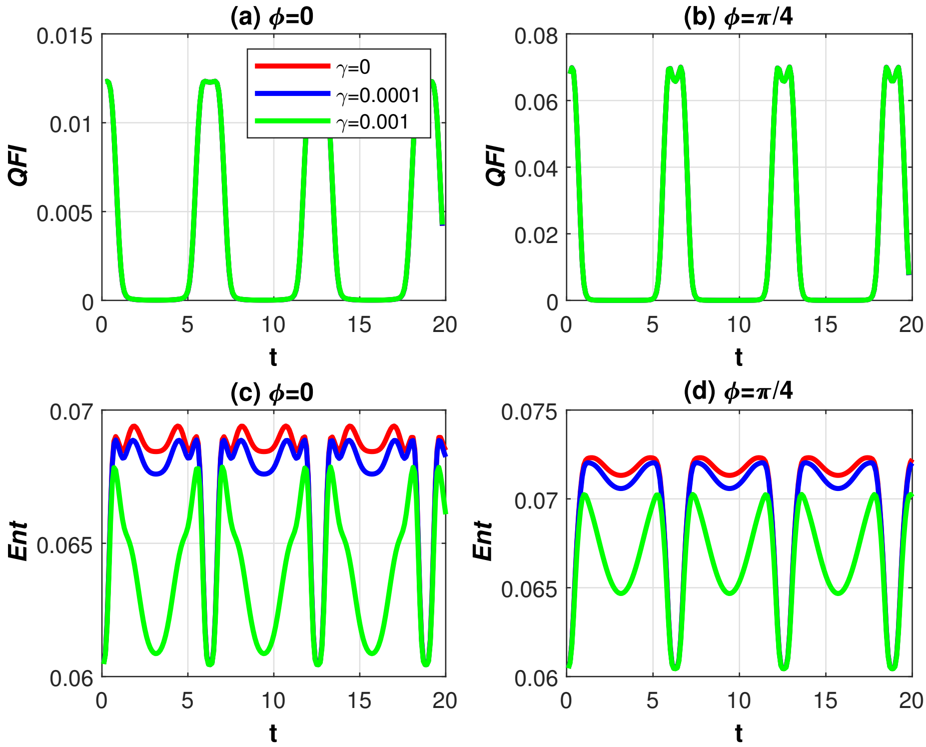

Figure 4. (Color online) The QFI (**upper** panel) and VNE (**lower** panel) as a function of time for two two-level atoms interacting with a thermal field in the presence ID parameter $\gamma = 0, 0.0001, 0.001$ and the phase shift estimator parameters $\phi = 0$ (**left** panel) and $\pi/4$ (**right** panel). The atomic motion parameter η is 1.

In Figures 5 and 6, we plot the QFI and VNE as a function of time for two two-level systems interacting with a coherent field for $|\alpha|^2 = 6$ and the phase shift parameters $\phi = 0$ (left panel) and $\pi/4$ (right panel) for different values of χ (non-linear Kerr medium) with and without atomic motion, i.e., $\eta = 0, 1$. The effect of the non-linear Kerr medium is found to be more prominent on the QE as compared to the QFI in the absence of atomic motion. It is seen that the non-linear Kerr medium plays a dominant role during the time evolution of the quantum system. The periodic behavior of QFI and QE is further suppressed under the effect of the non-linear Kerr medium. These results show the strong dependence of QFI and QE on the non-linear Kerr medium. It is observed that, at $\chi = 0.3$, t increases, but as time evolves it is saturated. In the case of $\chi = 1, 3$, VNE decreases, but this decrease become saturated as time evolves. Hence, it is concluded that at higher values of Kerr parameter, the QE decreases as compared to the lower values. However, the non-linear Kerr medium has no prominent effect on QFI at higher and lower values of Kerr parameter. In the presence of atomic motion, both QFI and VNE show periodic behavior. In Figures 7–10, we plot QFI and VNE as a function of time for two two-level atomic systems interacting with a coherent field for $|\alpha|^2 = 6$ and the phase shift parameters $\phi = 0$ (left panel) and $\pi/4$ (right panel) for different values of β (Stark shift) with and without atomic motion, i.e., $\eta = 0, 1$, respectively. It is found the Stark effect strongly influenced the entanglement of the two two-level atomic systems. It is observed that at $\beta = 0.3$, in the absence of atomic motion, the QFI decreases as time evolves, but VNE increases; however, at $\beta = 1, 3$, the periodic behavior of QFI and VNE is observed. Therefore, at higher values of β, the QE is sustaining but at lower values of β, it decreases. In the presence of atomic motion, QFI and VNE show a periodic response at different β values, so increasing β value in the presence of atomic motion does not effect the QE. Finally, the quantum system is found to be highly sensitive to these environmental influences.

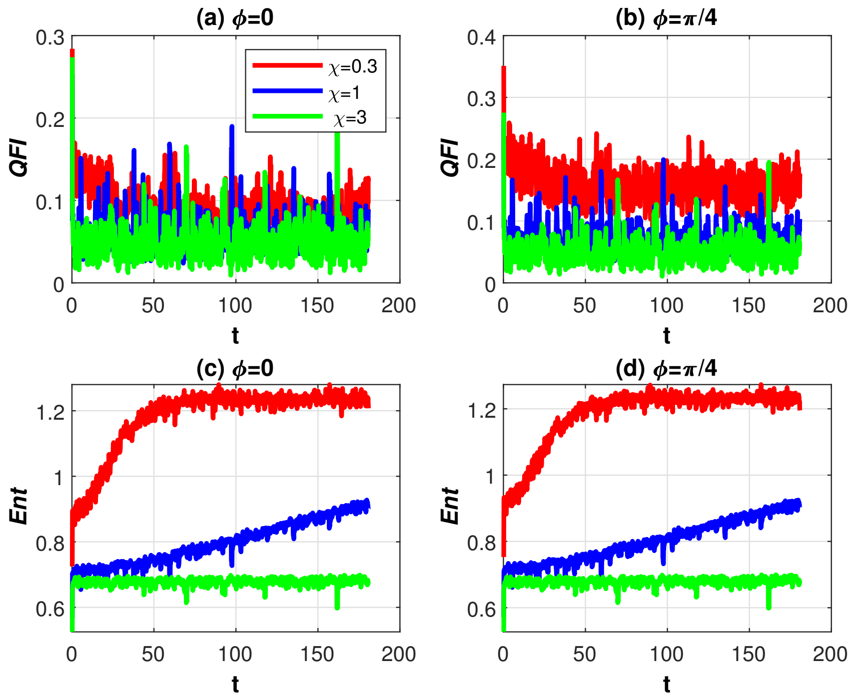

Figure 5. (Color online) The QFI (**upper** panel) and VNE (**lower** panel) as a function of time for a system of two two-level atoms having interaction with a coherent field for $|\alpha|^2 = 6$ and the phase shift estimator parameters $\phi = 0$ (**left** panel) and $\pi/4$ (**right** panel). The parameter η of atomic motion is ignored, and the value of $\chi = 0.3, 1, 3$ (non-linear Kerr).

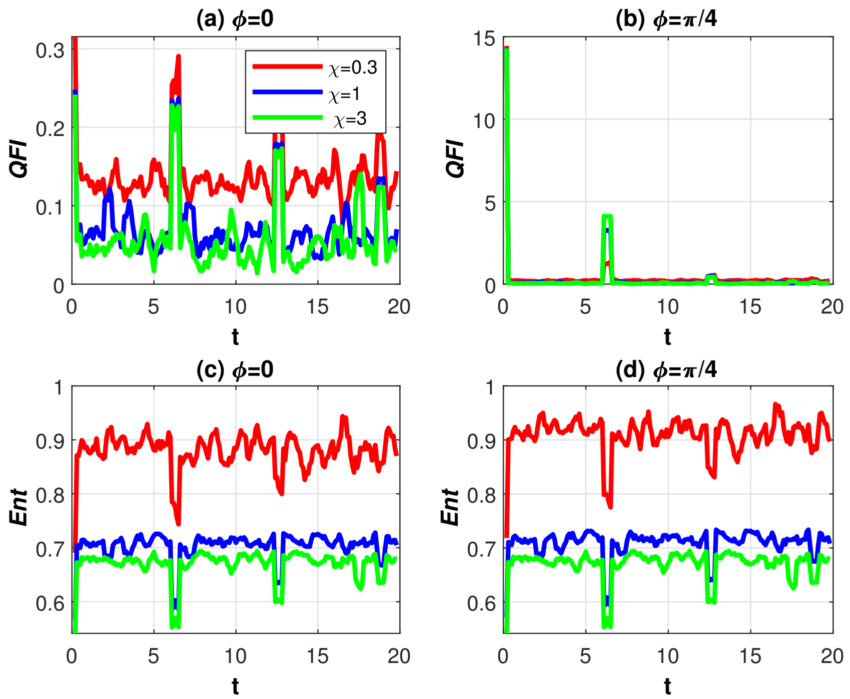

Figure 6. (Color online) The QFI (**upper** panel) and VNE (**lower** panel) as a function of time for a system of two two-level atoms having interaction with a coherent field for $|\alpha|^2 = 6$ and the phase shift estimator parameters $\phi = 0$ (**left** panel) and $\pi/4$ (**right** panel). The parameter η of atomic motion is 1, and the value of $\chi = 0.3, 1, 3$ (non-linear Kerr).

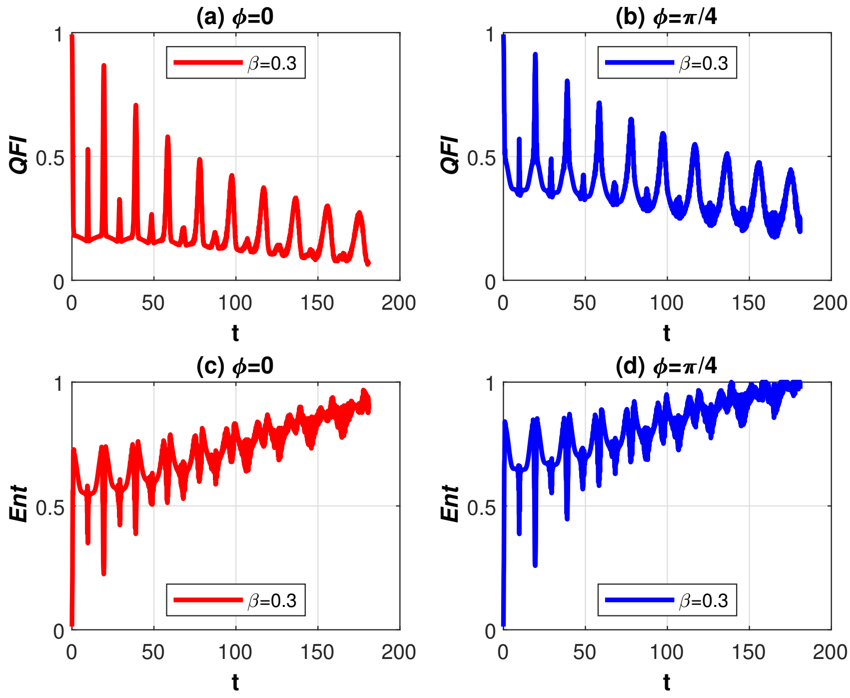

Figure 7. (Color online) The QFI (**upper** panel) and VNE (**lower** panel) as a function of time for a system of two two-level atoms having interaction with a coherent field for $|\alpha|^2 = 6$ and the phase shift estimator parameters $\phi = 0$ (**left** panel) and $\pi/4$ (**right** panel). The parameter η of atomic motion is ignored, and the value of $\beta = 0.3$ (Stark effect).

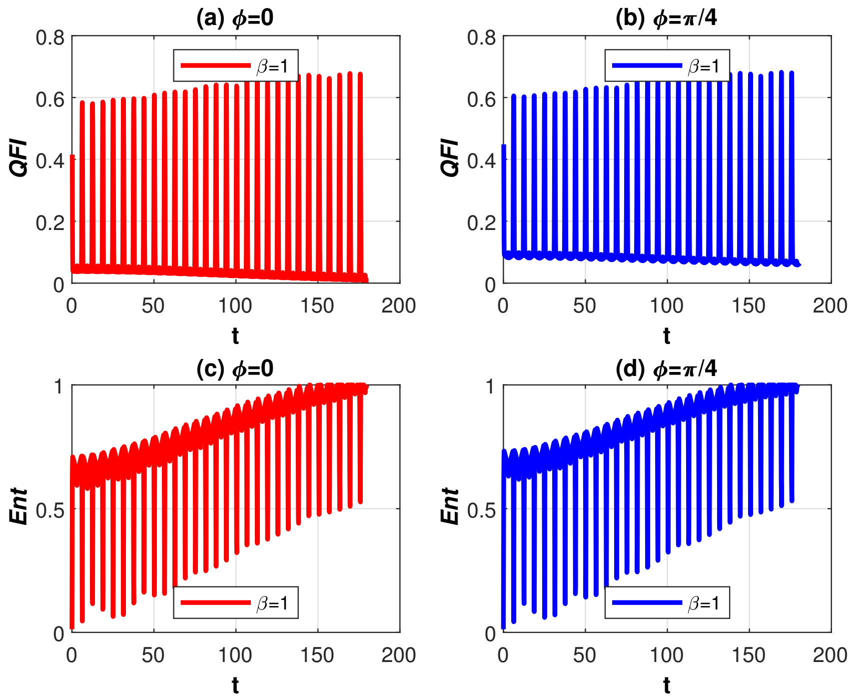

Figure 8. (Color online) The QFI (**upper** panel) and VNE (**lower** panel) as a function of time for a system of two two-level atoms having interaction with a coherent field for $|\alpha|^2 = 6$ and the phase shift estimator parameters $\phi = 0$ (**left** panel) and $\pi/4$ (**right** panel). The parameter η of atomic motion is ignored, and the value of $\beta = 1$ (Stark effect).

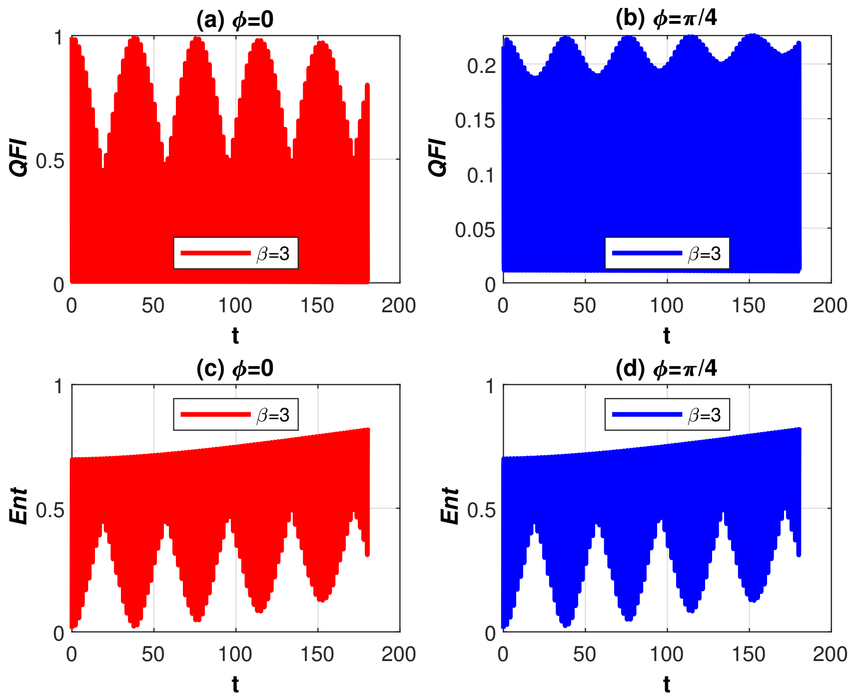

Figure 9. (Color online) The QFI (**upper** panel) and VNE (**lower** panel) as a function of time for a system of two two-level atoms having interaction with a coherent field for $|\alpha|^2 = 6$ and the phase shift estimator parameters $\phi = 0$ (**left** panel) and $\pi/4$ (**right** panel). The parameter η of atomic motion is ignored, and the value of $\beta = 3$ (Stark effect).

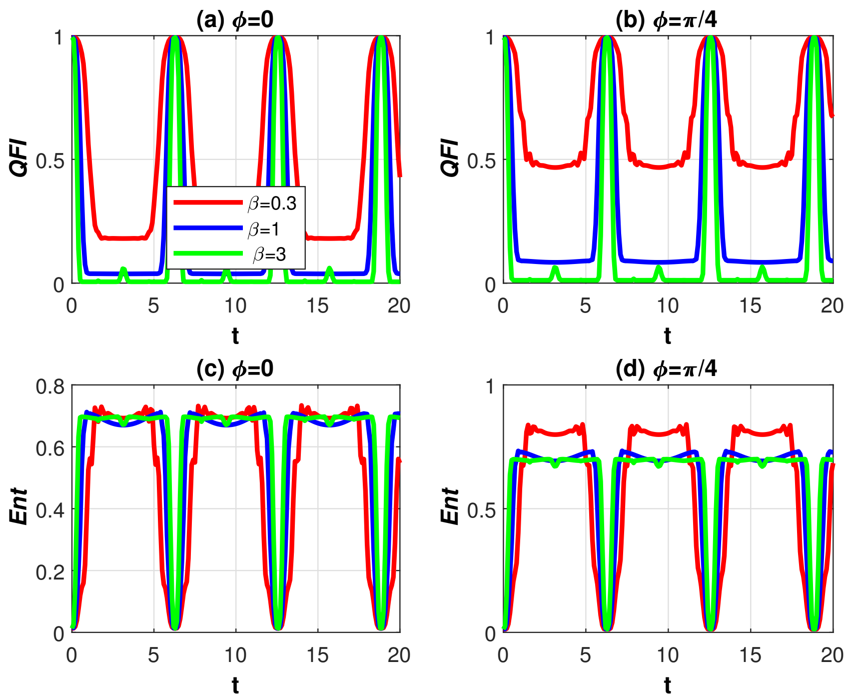

Figure 10. (Color online) The QFI (**upper** panel) and VNE (**lower** panel) as a function of time for a system of two two-level atoms having interaction with a coherent field for $|\alpha|^2 = 6$ and the phase shift estimator parameters $\phi = 0$ (**left** panel) and $\pi/4$ (**right** panel). The parameter η of atomic motion is 1, and the value of $\beta = 3$ (Stark effect).

6. Conclusions

In summary, we studied the dynamical evolution of QE and QFI for two moving two-level atoms interacting with coherent and thermal fields in the presence of ID. The time evolution of the entire system interacting with coherent and thermal environments is investigated numerically. It was seen that ID and a thermal environment play critical roles in the time evolution of the quantum system. Both QFI and VNE decreased as we increased the ID value in the absence of atomic motion, but showed periodic behavior in the presence of atomic motion. QFI and QE exhibited an opposite behavior in a thermal environment. QFI was found to be more prone to ID as compared to the entanglement in the presence of a thermal environment. QE drastically decreased when we increased the value of ID in the absence of atomic motion. Damping behavior of QE was observed at greater time-scales. The periodic behavior of QE due to atomic motion became modest under the environmental effects. The ID and thermal environment were found to suppress the non-classical effects of the quantum system. However, QE and QFI saturated to a lower level for longer time-scales under these environments in the absence of atomic motion.It is ought to be mentioned that the thermal environment induce decaying of the QE faster as compared to the decay induced by the ID when the atomic motion was not present. Based on [64], the results of our numerical calculations show that two two-level atomic systems are more robust against the variations in the chosen parameters. Finally, it was revealed that the presented system can be useful for generating and maintaining QE between the two two-level atoms in the presence of these environments.

We studied the dynamics of QE and QFI for two two-level atomic systems under the influence of a Stark shift and a non-linear Kerr medium. The time evolution of QFI and QE for two two-level atomic system influenced by the Stark effect and the non-linear Kerr-like medium was investigated. It was observed that the Stark effect and the non-linear Kerr medium play dominant roles during the time evolution of the quantum system. The effect of the non-linear Kerr medium was found to be more prominent on the QE as compared to the QFI in the absence of atomic motion. It was seen that the non-linear Kerr medium plays a dominant role during the time evolution of the quantum system. The periodic behavior of QFI and QE was further suppressed under the effect of the non-linear Kerr medium. These results show the strong dependence of QFI and QE on the non-linear Kerr medium. It was concluded that, at higher Kerr parameter values, QE decreases, as compared to lower values. However, the non-linear Kerr medium has no prominent effect on QFI, at either higher or lower Kerr parameter values. In the presence of atomic motion, both QFI and VNE show periodic behavior. Similarly, the Stark effect strongly influences the QE of two two-level atoms. In the absence of atomic motion, at higher β values, the QE is sustaining, but at smaller values, it is decreasing. In the presence of atomic motion, QFI and VNE show periodic response at different β values, so increasing β in the presence of atomic motion does not effect the QE. The QFI and QE evolves with time as we increase the Stark effect parameter. Finally, the quantum system was found to be highly sensitive to these environmental influences.

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