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Yang–Mills Theory of Gravity

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Abstract: The canonical formulation of general relativity (GR) is based on decomposition space–time manifold M into $R \times \Sigma$, where R represents the time, and Σ is the three-dimensional space-like surface. This decomposition has to preserve the invariance of GR, invariance under general coordinates, and local Lorentz transformations. These symmetries are associated with conserved currents that are coupled to gravity. These symmetries are studied on a three dimensional space-like hypersurface Σ embedded in a four-dimensional space–time manifold. This implies continuous symmetries and conserved currents by Noether's theorem on that surface. We construct a three-form $E_i \wedge DA^i$ (D represents covariant exterior derivative) in the phase space (E_i^a, A_a^i) on the surface Σ, and derive an equation of continuity on that surface, and search for canonical relations and a Lagrangian that correspond to the same equation of continuity according to the canonical field theory. We find that Σ_i^{0a} is a conjugate momentum of A_a^i and $\Sigma_i^{ab} F_{ab}^i$ is its energy density. We show that there is conserved spin current that couples to A^i, and show that we have to include the term $F_{\mu\nu i} F^{\mu\nu i}$ in GR. Lagrangian, where $F^i = DA^i$, and A^i is complex $SO(3)$ connection. The term $F_{\mu\nu i} F^{\mu\nu i}$ includes one variable, A^i, similar to Yang–Mills gauge theory. Finally we couple the connection A^i to a left-handed spinor field ψ, and find the corresponding beta function.

Keywords: gravitational Lagrangian; complex connection A^i; left-handed spinor field

1. Introduction

Gravity can be formulated based on gauge theory by gauging the Lorentz group $SO(3,1)$ [1]. For this purpose, we need to fix some base space and consider that the Lorentz group $SO(3,1)$ acts locally on Lorentz frames which are regarded as a frame bundle over a fixed base space. We can consider this base space as an arbitrary space–time manifold M with coordinates x^μ, and consider the local Lorentz frame as an element in the tangent frame bundle over M. By that we have two symmetries; invariance under continuous transformations of local Lorentz frame, $SO(3,1)$ group, and invariance under diffeomorphism of the space–time M, which is originally considered as a base space [2].

Since the group $SO(3)$ is a subgroup of $SO(3,1)$, the Lagrangian of gravity has an internal gauge symmetry group $SO(3)$. Thus the elements of $SO(3)$ act locally on some spacial Lorentz orthonormal frames (e^1, e^2, e^3), we consider these frames as a basis of the tangent vector bundle on the three-dimensional space-like hypersurface Σ. Using some local coordinate system σ^a, $a = 1, 2, 3$ on Σ, we expand these basis vectors into $e^i = e_a^i d\sigma^a$, so defining the gravitational field e_a^i with metric $g_{ab} = \delta_{ij} e_a^i e_b^j$ on Σ. Using the isomorphism between Lie algebra of $SU(2)$ and that of $SO(3)$, one can regard e_a^i also as a local $su(2)$-valued one-form. So we have an $SU(2)$ vector bundle with real spin connection ω^{ij} [3]. These facts can be generated into self-dual and anti-self-dual formalism of general relativity (GR) with complex connection A^i and complex conjugate one-form field E_i.

This paper proceeds as follows: We start with the four-dimensional (4D) Palatini Lagrangian and perform a $3+1$ decomposition based on the decomposition $M \rightarrow R \times \Sigma$, where R represents the time, and Σ is the three-dimensional (3D) space-like surface, thus we specify the Lagrangian part $L_1(g_{ab})$ on Σ. Then we try to show that $L_1(g_{ab})$ is independent of time on this surface, we try to prove this fact using the fact that there are no dynamics in the space-like region of M. After that we use $\frac{d}{dt}L_1(g_{ab}) = 0$ to get an equation of continuity on Σ, and search for canonical relations and Lagrangian that correspond to the same equation of continuity according to the canonical field theory. We find that Σ_i^{0a} is a conjugate momentum of A_a^i and $\Sigma_i^{ab}F_{ab}^i$ is its energy density. We obtain a Lagrangian for the connection A^i in 4-manifold M, then we couple it to a left-handed fermion field and find the beta function.

2. Decomposition Space–Time Manifold M into $R \times \Sigma$

The formulation of GR based on decomposition space–time manifold M into $R \times \Sigma$ is needed for expressing the metric of space–time as a solution of an equation for time evolution, such as in the Hamiltonian formulation. Thus the time evolution is the changing of the geometry of this surface. This decomposition preserves the continuous symmetries (gauge invariance and diffeomorphism invariance) of GR and its canonical quantization, so we can use it for the gauge theory of GR [4–7].

We define gravitational field as a one-form $e^I = e_\mu^I(x)dx^\mu$ that is related with metric $g_{\mu\nu}(x)$ on an arbitrary space–time manifold M by $g_{\mu\nu} = \eta_{IJ}e_\mu^I e_\nu^J$, with spin connection $\omega^{IJ}(x) \in \Omega^1(M, so(3,1))$, where $so(3,1)$ is Lie algebra of Lorentz group $SO(3,1)$. The spin connection defines covariant derivative D_μ that acts on all fields which have Lorentz indices $(I, J, ...)$:

$$D_\mu v^I = \partial_\mu v^I + \omega_{\mu J}^I v^J.$$

We start with the GR Lagrangian of the form

$$L(e,\omega) = (16\pi G)^{-1} e_I^\mu e_J^\nu (R_{\mu\nu})^{IJ} e, \tag{1}$$

where

$$R^{IJ} = d\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}$$

is the Riemannian curvature tensor and e_I^μ satisfies $e_\mu^I e_J^\mu = \delta_J^I$.

By the decomposition $M \rightarrow R \times \Sigma$, we decompose this Lagrangian into

$$L(e,\omega) = (16\pi G)^{-1} e_i^a e_j^b (R_{ab})^{ij} e + (16\pi G)^{-1} e_0^a e_J^b (R_{ab})^{0J} e + (16\pi G)^{-1} e_I^0 e_J^a (R_{0a})^{IJ} e, \tag{2}$$

where i and j are Lorentz indices for $I = i = 1,2,3$, and $a = 1,2,3$. The part

$$L_1 = (16\pi G)^{-1} e_i^a e_j^b (R_{ab})^{ij} e$$

has the gauge symmetry of the group $SO(3)$, which is a subgroup of $SO(3,1)$, it also relates to the geometry of the surface $\Sigma(\sigma^a)$ under the variation in the direction of $\Sigma(\sigma^a)$ subject to $\delta e_0^a = 0$, $\delta\omega_0^{ij} = 0$, since it depends only on the metric $g_{ab} = \delta_{ij}e_a^i e_b^j$ which is defined on $\Sigma(\sigma^a)$, which is intrinsic geometry.

Since $(R_{ab})^{ij}$ is an anti-symmetric tensor, we can introduce a new one-form field E_c^k, the Hodge dual of $e^i \wedge e^j$ in the internal spin space on the surface $\Sigma(\sigma^a)$; $E^i = E_a^i d\sigma^a$, it is called the gravitational electric field [8,9]

$$\frac{1}{2}\left(e_i^a e_j^b - e_j^a e_i^b\right) = \varepsilon^{abc}\varepsilon_{ijk}E_c^k.$$

In self-dual formalism of GR, E^a_i is complex given by [10,11]

$$E^{ia} = \frac{1}{2}\varepsilon^{abc} P^i_{IJ} e^I_b e^J_c, \text{ with } E^i_{ab} = \Sigma^i_{ab} = P^i_{IJ} e^I_a e^J_b \tag{3}$$

where P^i_{IJ} is a self-dual projector given by

$$P^i_{IJ} = \frac{1}{2}\varepsilon^i_{jk}, \text{ for } I = i, J = j, \text{ and } P^i_{0j} = -P^i_{j0} = \frac{i}{2}\delta^i_j, \text{ for } I = 0, J = j \neq 0. \tag{4}$$

For example $E^{1a} = \frac{1}{2}\varepsilon^{abc}(e^2_b e^3_c + ie^0_b e^1_c)$.

For the Lagrangian part L_1 on $\Sigma(\sigma^a)$, we use the first one:

$$L_1 = (16\pi G)^{-1} E^k_c \varepsilon^{abc} \varepsilon_{ijk} (R_{ab})^{ij} e. \tag{5}$$

The remaining part

$$L_2 = (16\pi G)^{-1} e^a_0 e^b_J (R_{ab})^{0J} e + (16\pi G)^{-1} e^0_I e^a_J (R_{0a})^{IJ} e,$$

associates with the time evolution under the variation in the normal direction of $\Sigma(\sigma^a)$, it is subject to $\delta e^a_i = 0$, $\delta\omega^{ij}_a = 0$, and changes the geometry of the surface $\Sigma(\sigma^a)$ during the time. Dynamics, such as propagation $g_{ab}(x) \to g_{ab}(x')$ in the time-like region $(\Delta x)^2 = g_{\mu\nu}(x)\Delta x^\mu \Delta x^\nu < 0$, and determines how the surface $\Sigma(\sigma^a)$ is embedded into the $4D$ manifold M, which is extrinsic geometry. But the surface $\Sigma(\sigma^a)$ is embedded in a space-like region in M; $V_1, V_2 \in T_p\Sigma(\sigma^a), g(V_1, V_2) > 0$, so there are no dynamics on $T_p\Sigma(\sigma^a)$. We can see this fact by noting that $\nabla_t g_{ab} = 0$ (the covariant derivative of the metric is zero), so

$$\frac{dL_1(g_{ab})}{dt} dt = \frac{\partial L_1(g_{ab})}{\partial g_{ab}} \nabla_t g_{ab} dt = 0.$$

We will rewrite $\nabla_t g_{ab} = 0$ as $t^a t^b \nabla_t g_{ab} = 0$, for $t^a \in T_p\Sigma_t(\sigma^a)$. The formula $t^a t^b \nabla_t g_{ab} = 0$ is more general than $\nabla_t g_{ab} = 0$ since there is $\nabla_t g_{ab} \notin \Gamma\left(T^*\Sigma_t(\sigma^a) \times T^*\Sigma_t(\sigma^a)\right)$, and so its projection onto $T_p\Sigma(\sigma^a)$ is zero. This case appears in the diffeomorphism maps of $\Sigma(\sigma^a)$ into another space-like surface, as we will see.

We can study the embedding of $\Sigma(\sigma^a)$ by letting the time derivative of a position vector on its tangent space be in the direction of the normal to this tangent space $T_p\Sigma(\sigma^a)$. We can see this by considering a position vector $t^a \in T_p\Sigma_t(\sigma^a)$ that satisfies $\dot{t}^a t_a = \dot{t}_a t^a = 0$. Let $\dot{t}^a \in N_p\Sigma_t(\sigma^a)$, where $N_p\Sigma_t(\sigma^a)$ is the normal space to $\Sigma_t(\sigma^a) = \{t\} \times \Sigma(\sigma^a)$, and $\dot{t}^a = \nabla_t t^a$ is a covariant derivative of t^a with respect to the time $t \in R \subset R \times \Sigma(\sigma^a)$. From $t_a = g_{ab}t^b$, we obtain $\dot{t}_a = \dot{g}_{ab}t^b + g_{ab}\dot{t}^b$, so $t^a\dot{t}_a = t^a\dot{g}_{ab}t^b + t^a g_{ab}\dot{t}^b$. Since $\dot{t}^a \in N_p\Sigma_t(\sigma^a)$, we have $t^a g_{ab}\dot{t}^b = 0$, thus we get $\dot{g}_{ab}t^a t^b = t^a t^b \nabla_t g_{ab} = 0$ which means that the points of M do not expand nor contract covariantly in the space-like $T_p\Sigma_t(\sigma^a)$, but $t^a t^b \partial_t g_{ab} \neq 0$ is possible.

Therefore, for a diffeomorphism map of $\Sigma(\sigma^a)$ into another space-like surface (consider this as time evolution), it must be

$$\nabla_t(g_{ab}) \notin \Gamma\left(T^*\Sigma_t(\sigma^a) \times T^*\Sigma_t(\sigma^a)\right),$$

and

$$\nabla_t(g_{ab}) \in \Gamma(N^*\Sigma_t(\sigma^a) \times T^*\Sigma_t(\sigma^a)) \oplus \Gamma(N^*\Sigma_t(\sigma^a) \times N^*\Sigma_t(\sigma^a)).$$

This means for each $x, x' \in \Sigma_t(\sigma^a)$, the covariant propagation $g_{ab}(x) \to g_{ab}(x') \equiv g_{ab}(x) + \xi^\mu \nabla_\mu g_{ab}(x)$ does not occur in $T^*_p\Sigma_t(\sigma^a) \times T^*_p\Sigma_t(\sigma^a)$, but it occurs in

$$N^*_p\Sigma_t(\sigma^a) \times T^*_p\Sigma_t(\sigma^a) \oplus N^*_p\Sigma_t(\sigma^a) \times N^*_p\Sigma_t(\sigma^a).$$

This is a changing in the embedding of $\Sigma(\sigma^a)$ in M. Let $t_a \in T^*_p\Sigma_t(\sigma^a)$ satisfy $(\dot{t}_a) = n$, where $(\cdot)$ is matrix notation and n is the normal to $\Sigma_t(\sigma^a)$. This normal is in the direction of the time, so it carries one index; $n = n^0$, thus $(\dot{t}_a) = n_0$. It must also satisfy $(\dot{t}^a) \notin T_p\Sigma_t(\sigma^a)$, so $(g_{ab})(\dot{t}^b) = 0$. Using this in $(\dot{t}_a) = (\dot{g}_{ab})(t^b) + (g_{ab})(\dot{t}^b)$, we obtain $(\dot{t}_a) = (\dot{g}_{ab})(t^b)$, so we get $n_0 = (\dot{g}_{ab})_{0b}t^b$. Thus we get $n_0 n^0 = n^0(\dot{g}_{ab})_{0b}t^b$, and by $n_0 n^0 = 1$, we obtain $n^0(\dot{g}_{ab})_{0b}t^b = 1$. This formula is for determining $(\dot{g}_{ab}) = \nabla_t(g_{ab}) \in \Gamma(N^*\Sigma_t(\sigma^a) \times T^*\Sigma_t(\sigma^a))$.

Let us suggest a formula for determining $(\dot{g}_{ab}) \in \Gamma(N^*\Sigma_t(\sigma^a) \times T^*\Sigma_t(\sigma^a))$ like

$$\nabla_t g_{ab} = f^c_{ab} g_{0c}, \quad f^c_{ab} = f^c_{ba}, \quad t^a t^b f^c_{ab} = 0,$$

where $g_{0c} \in \Gamma(N^*\Sigma_t(\sigma^a) \times T^*\Sigma_t(\sigma^a))$, and $g_{\mu\nu} = (g_{00}, g_{0a}, g_{ab})$ is the full metric. Let us write f^a_{bc} using matrix notation f^a, its elements are $(f^a)_{bc} = f^a_{bc}$. Let $\tilde{f}_a$ satisfy $(\tilde{f}_a)^{bc}(f^{a'})_{bc} = \delta^{a'}_a$. Thus we get the inversion

$$(\tilde{f}_c)^{ab}\nabla_t g_{ab} = (\tilde{f}_c)^{ab}(f^{c'})_{ab} g_{0c'} = \delta^{c'}_c g_{0c'} = g_{0c}, \tag{6}$$

this is obtaining the metric component g_{0a} from $\nabla_t g_{ab}$; the changing of the metric g_{ab} with respect to the time. Thus if we fix the metric component g_{00}, like $g_{00} = -1$, we obtain the map $g_{ab}(t) \to g_{\mu\nu} = (-1, g_{0a}, g_{ab})$. So we have an immersion $\Gamma(T^*\Sigma(\sigma^a)) \to \Gamma(N^*\Sigma_t(\sigma^a) \times T^*\Sigma_t(\sigma^a)) \subset \Gamma(T^*M)$. In another words, for an immersion $M^{(n)} \to M^{(n+1)}$, and under some hypotheses, we can construct a metric on $M^{(n+1)}$ using the metric on M^n. Our hypothesis here is $t^a t^b \nabla_t g_{ab} = 0$.

First we find the matrices f^a then $\tilde{f}_a$. The matrices f^a are symmetric and satisfy $t^b t^c (f^a)_{bc} = t^\top (f^a) t = 0$, where the vector $t^\top = (t_1, t_2, t_3)_p$ is the unit vector in $T_p\Sigma_t(\sigma^a))$, we can write them in a simple form like

$$f^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos(\theta_1) & \sin(\theta_1) \\ 0 & \sin(\theta_1) & -\cos(\theta_1) \end{pmatrix}, \quad f^2 = \begin{pmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) \\ 0 & 0 & 0 \\ \sin(\theta_2) & 0 & -\cos(\theta_2) \end{pmatrix},$$

$$f^3 = \begin{pmatrix} \cos(\theta_3) & \sin(\theta_3) & 0 \\ \sin(\theta_3) & -\cos(\theta_3) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the angles θ_1, θ_2 and θ_3 can be determined to satisfy $t^a t^b (f^c)_{ab} = 0$, thus we obtain

$$\tan(\theta_1) = \frac{t_3^2 - t_2^2}{2t_3 t_2}, \quad \tan(\theta_2) = \frac{t_3^2 - t_1^2}{2t_3 t_1}, \quad \tan(\theta_3) = \frac{t_2^2 - t_1^2}{2t_2 t_1}.$$

Therefore the matrices $\tilde{f}_a$ can be written in the form

$$\tilde{f}_1 = \begin{pmatrix} 0 & c_1\cos(\theta_3) & 0 \\ c_1\cos(\theta_3) & 2c_1\sin(\theta_3) & \sin(\theta_1) \\ 0 & \sin(\theta_1) & 0 \end{pmatrix}, \quad \tilde{f}_2 = \begin{pmatrix} -2c_1\sin(\theta_3) & c_1\cos(\theta_3) & \cos(\theta_3) \\ c_1\cos(\theta_3) & 0 & c_2\cos(\theta_1) \\ \cos(\theta_3) & c_2\cos(\theta_1) & 2c_2\sin(\theta_1) \end{pmatrix}$$

$$\tilde{f}_3 = \begin{pmatrix} 0 & \cos(\theta_3) & 0 \\ \cos(\theta_3) & -2c_3\sin(\theta_1) & c_3\cos(\theta_1) \\ 0 & c_3\cos(\theta_1) & 0 \end{pmatrix},$$

the constants c_1, c_2 and c_3 are determined to satisfy

$$(\tilde{f}_1)^{ab}(f^1)_{ab} = (\tilde{f}_2)^{ab}(f^2)_{ab} = (\tilde{f}_3)^{ab}(f^2)_{ab} = 1.$$

We can consider that as a continuous changing in the embedding of $\Sigma_t(\sigma^a)$ in M;which is a diffeomorphism map. Since the Lagrangian L_1 depends only on g_{ab} while the term L_2 depends on (g_{00}, g_{0a}, g_{ab}), thus by the previous discussing we have a map $L_1(g_{ab}) \to L_2(g_{00}, g_{0a}, g_{ab})$ (time evolution), with $g_{00} = -1, g_{0a} = (\tilde{f}_a)^{bc}\nabla_t g_{bc}$. As illustrated in the following Figure 1.

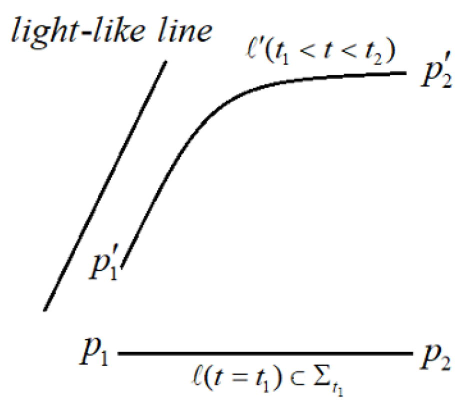
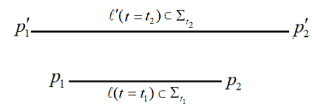


Figure 1. Propagation of space-like surface.

Let $\ell \subset \Sigma_{t_1}$ which propagated to $\ell' \subset \Sigma_{t_1<t<t_2}$, both surfaces are space-like, but the trajectories of the propagation $p_1 \to p'_1$ and $p_2 \to p'_2$ lie in the time-like region. The length of ℓ is also increased. Thus, this means that the $3D$ metric g_{ab} mapped to $4D$ metric (g_{00}, g_{0a}, g_{ab}) by Equation (6), which means the length ℓ' is given by

$$\ell'^2 = \int_{p'_1}^{p'_2} \left(-dtdt + g_{0a}dtd\sigma^a + g_{ab}d\sigma^a d\sigma^b\right),$$

which again can be mapped into a space-like surface Σ_{t_2} by an isometric map such $-dtdt + g_{0a}dtd\sigma^a + g_{ab}d\sigma^a d\sigma^b = g'_{a'b'}d\sigma^{a'}d\sigma^{b'}$. which results in Figure 2:



$$\ell'^2 - \ell^2 = \int_{p_1'}^{p_2'} (-dtdt + g_{0a}dtd\sigma^a).$$

Figure 2. Changing distance between two space-like points after propagation.

$$\ell'^2 - \ell^2 = \int_{p_1'}^{p_2'} (-dtdt + g_{0a}dtd\sigma^a).$$

The limit $\ell'^2 - \ell^2 = 0$ corresponds to $dt = 0$ and $d\sigma^a = 0$, not to $-dtdt + g_{0a}dtd\sigma^a = 0$ (with $dt \neq 0$ and $d\sigma^a \neq 0$). For example, if two particles P_1 and P_2 exchange photons with wavelength $\lambda = \ell \approx \sigma_2^a - \sigma_1^a$ in $t = t_1$, then in $t = t_2$ they measure a different wavelength, namely $\lambda + \Delta\lambda = \ell'$. The difference is given by $2\lambda\Delta\lambda = -\lambda^2 + g_{0a}\lambda^2$, where we set $dt = d\sigma^a = \lambda$. Thus the two particles P_1 and P_2 measure after a time $dt = \lambda$ the difference $2\Delta\lambda/\lambda = -1 + g_{0a}$, where g_{0a} is given in Equation (6).

Thus we study the embedding of $3D$ surface Σ_t and its changing in $4D$ manifold M using the $3D$ metric g_{ab} on Σ_t and its derivative with respect to the time.

The Lagrangian $L_1(g_{ab})$ is a function on the space-like space $T_p^*\Sigma_t(\sigma^a)$, therefore, it is independent of time, $\frac{d}{dt}L_1(g_{ab}) = 0$ on $T_p^*\Sigma_t(\sigma^a)$. We can see this by using the fact

$$\nabla_t(g_{ab}) \in \Gamma(N^*\Sigma_t(\sigma^a) \times T^*\Sigma_t(\sigma^a)) \oplus \Gamma(N^*\Sigma_t(\sigma^a) \times N^*\Sigma_t(\sigma^a)),$$

we have

$$L_1(g_{ab}(t+dt)) - L_1(g_{ab}(t)) = \frac{d}{dt}L_1(g_{ab}(t))dt = \frac{\partial L_1(g_{ab})}{\partial g_{ab}}\nabla_t(g_{ab})dt,$$

which is a function on $N_p^*\Sigma_t(\sigma^a) \times T_p^*\Sigma_t(\sigma^a) \oplus N_p^*\Sigma_t(\sigma^a) \times N_p^*\Sigma_t(\sigma^a)$, not on $T_p^*\Sigma_t(\sigma^a)$; which means there are no dynamics on the space-like region $(\Delta x)^2 > 0$.

Let us write $dt\partial_t L_1(g_{ab}) = (\Sigma(\sigma^a), d\theta)$, where (Σ, V) is a projection of $V \in \wedge^4 T_p^* M$ onto a surface Σ, the inner product of V with tangent basis in $T_p\Sigma$, defined below in Equation (17), and θ is three-form in the phase space (E_i, ω^{ij}) on $\Sigma(\sigma^a)$. Thus we write

$$\left(\Sigma(\sigma^a), dt \wedge \frac{d}{dt}\theta\right) = dt\frac{d}{dt}L_1(g_{ab}) = 0.$$

If we write θ as

$$\theta(E, \omega, \Sigma_t(\sigma^a)) = (16\pi G)^{-1/2}\varepsilon_{ijk}E^k \wedge R^{ij}, \tag{7}$$

with $R^{ij} = (R_{ab})^{ij} d\sigma^a \wedge d\sigma^b$ and $E^i = E_a^i d\sigma^a$, we obtain

$$\left(\Sigma(\sigma^a), dt \wedge \frac{d}{dt}\theta\right) = (16\pi G)^{-1/2} dt \frac{d}{dt}(E_c^k \varepsilon^{abc}\varepsilon_{ijk}(R_{ab})^{ij} e)$$
$$= (16\pi G)^{1/2} dt \frac{d}{dt} L_1 = 0.$$

Since θ is three-form on $\Sigma(\sigma^a)$, so $d\sigma^a \wedge \frac{\partial}{\partial\sigma^a}\theta = 0$, if we add it to the last formula, we get

$$\left(\Sigma(\sigma^a), dt \wedge \frac{\partial}{\partial t}\theta + d\sigma^a \wedge \frac{\partial}{\partial \sigma^a}\theta\right) = (\Sigma(\sigma^a), d\theta) = 0.$$

This relates the Lagrangian L_1 and θ with this surface. Under arbitrary transformation $(t, \sigma^a) \to x^\mu$, the two Lagrangian parts L_1 and L_2 will mix. Thus $d\sigma^a = \frac{\partial\sigma^a}{\partial x^\mu}dx^\mu$ and the basis on Σ transforms as

$$\partial_a \wedge \partial_b \wedge \partial_c \to \frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b}\frac{\partial x^\rho}{\partial\sigma^c}\partial_\mu \wedge \partial_\nu \wedge \partial_\rho. \tag{8}$$

Therefore, the components of three-form θ transforms as

$$\theta_{abc} \to \theta_{\mu\nu\rho} = \theta_{abc}\frac{\partial\sigma^a}{\partial x^\mu}\frac{\partial\sigma^b}{\partial x^\nu}\frac{\partial\sigma^c}{\partial x^\rho}.$$

To keep the invariance under this transformation, that is $(\Sigma, d\theta) = 0$ still holds, we write the three-form θ in the phase space (E_i, ω^{ij}) on M, and let $(\Sigma, d\theta)$ be its projection onto Σ. Therefore we write

$$\theta(E, \omega) = (16\pi G)^{-1/2}\varepsilon_{ijk}E^k \wedge R^{ij}, \tag{9}$$

thus we get three-form θ in the phase space (E_i, ω^{ij}) on M, it has internal $SO(3)$ symmetry. Its projection onto $\Sigma_t(\sigma^a)$ is

$$(\Sigma_t(\sigma^a), \theta(E, \omega)) = (16\pi G)^{-1/2}E^k_c\varepsilon^{abc}\varepsilon_{ijk}R^{ij}_{ab}.$$

We write θ using self-dual formalism (Plebanski formalism) [11] in which the connection A^i is a three-complex one-form given by

$$\omega^{IJ} \to A^i = P^i{}_{IJ}\omega^{IJ}, \tag{10}$$

where P^i_{IJ} is self-dual projector given by Equation (3). The curvature which associates with this connection is

$$F^i = dA^i + \varepsilon^i{}_{jk}A^j \wedge A^k.$$

On the surface $\Sigma_t(\sigma^a)$, it is

$$F^i_{ab} = \frac{1}{2}\left(\partial_b A^i_b - \partial_b A^i_a + \varepsilon^i{}_{jk}A^j_a A^k_b\right).$$

Also

$$e^{[a}_I e^{b]}_J \to P^{IJ}_i e^{[a}_I e^{b]}_J = E^{ab}_i = \varepsilon^{abc}E_{ic}. \tag{11}$$

Thus we have self-dual plus anti-self-dual projection:

$$e^a_I e^b_J R(\omega)^{IJ}_{ab} \to \varepsilon^{abc}E_{ia}F^i_{bc} + \varepsilon^{abc}\bar{E}_{ia}\bar{F}^i_{bc}, \tag{12}$$

where $\bar{E}_{ia}$ and $\bar{F}^i_{bc}$ are the Hermitian conjugate of E_{ia} and F^i_{bc}. This projection relates to the decomposition of Lie algebra of the Lorentz group $SO(1,3)$ into two copies of Lie algebra of $SL(2,R)$ [12].

We write the components of the curvature as

$$F^i_{ab} = \frac{1}{2}\left(D_a A^i_b - D_b A^i_a\right) = \frac{1}{2}\left(\partial_a A^i_b - \partial_b A^i_a + \varepsilon^i{}_{jk}A^j_a A^k_b\right), \tag{13}$$

where
$$D_a A_b^i = \partial_a A_b^i + \frac{1}{2}\varepsilon^i{}_{jk} A_a^j A_b^k,$$
which motivates introducing a notation of the covariant derivative like [7]
$$DV^i = dV^i + \frac{1}{2}\varepsilon^i{}_{jk} A^j \wedge V^k,$$
or
$$D_\mu V_\nu^i = \partial_\mu V_\nu^i + \frac{1}{2}\varepsilon^i{}_{jk} A_\mu^j V_\nu^k = \partial_\mu V_\nu^i - \frac{i}{2} A_\mu^j (T_A^j)^{ik} V_\nu^k,$$
so
$$D_\mu = \partial_\mu - \frac{i}{2} A_\mu^j (T_A^j),$$
where the matrix elements $(T_A^j)^{ik} = -i\varepsilon^{jik}$ are the elements of the generators T_A^j in the adjoint representation of the group $SU(2)$ [13]. The coupling constant here is $g = 1$. In general we write this covariant derivative as
$$D_\mu = \partial_\mu - \frac{i}{2} g A_\mu^j (T_A^j). \tag{14}$$

Using the projection in Equation (12), we rewrite θ in the complex phase space (E_i^a, A_a^i) as
$$\theta(E, \omega, \Sigma(\sigma^a)) = (16\pi G)^{-1/2} \varepsilon^{abc} E_{ic} F_{ab}^i(A) e d^3\sigma,$$
where we take in consideration only the first part, the second is obtained by taking the Hermitian conjugate. We can write it as three-form on M as done in Equation (9), we obtain
$$\theta(E, A) = (16\pi G)^{-1/2} E_i \wedge DA^i. \tag{15}$$

Its projection onto $\Sigma_t(\sigma^a)$ is
$$(\Sigma_t(\sigma^a), \theta) = (16\pi G)^{-1/2} \varepsilon^{abc} E_{ia} F_{bc}^i(A),$$
where A^i is complex $SO(3)$ connection, and E_i is a complex one-form as defined in Equation (11).

3. Equation of Continuity on the Hypersurface $\Sigma_t(\sigma^a)$

We have showed that the Lagrangian $L_1(g_{ab})$ is independent of time, $\frac{d}{dt} L_1(g_{ab}) = 0$ on $\Sigma(\sigma^a)$ since $\nabla_t(g_{ab}) \notin \Gamma(T^*\Sigma_t(\sigma^a) \times T^*\Sigma_t(\sigma^a))$. This relates to the fact that there are no dynamics in the space-like $T\Sigma_t(\sigma^a))$; the points of M do not expand nor contract covariantly in this region, $\nabla_\mu t^a \neq 0$. Note that although $\nabla_\mu t^a = 0$, but it may be $\partial_\mu t^a \neq 0$. We had
$$\nabla_t(g_{ab}) \in \Gamma(N^*\Sigma_t(\sigma^a) \times T^*\Sigma_t(\sigma^a)) \oplus \Gamma(N^*\Sigma_t(\sigma^a) \times N^*\Sigma_t(\sigma^a)),$$
which shows that the two parts of the Lagrangian L_1 and L_2 mix by time evolution. Then we wrote $dt \frac{d}{dt} L_1(g_{ab}) = 0$ as $(\Sigma(\sigma^a), d\theta) = 0$, where θ is three-form in the phase space (E_i, ω^{ij}) on M (Equation (7)), its projection onto $\Sigma_t(\sigma^a)$ is Equation (17)
$$(\Sigma_t(\sigma^a), \theta(E, \omega)) = (16\pi G)^{-1/2} \varepsilon_{ijk} \varepsilon^{abc} E_a^i R_{bc}^{jk}(\omega).$$

In self-dual formalism, we obtained $\theta(E,A) = (16\pi G)^{-1/2} E_i \wedge DA^i$, with

$$(\Sigma_t(\sigma^a), \theta(E,A)) = (16\pi G)^{-1/2} \varepsilon^{abc} E_{ia} F^i_{bc}(A).$$

Our condition $(\Sigma(\sigma^a), d\theta) = 0$ makes sense here because of the decomposition $R \times \Sigma$ and fixing a coordinate system σ^a on the hypersurface Σ, this yields to an equation of continuity on this surface. For this purpose, we take the inner product of the four-form $d\theta$ with a tangent basis on the surface $\Sigma_t(\sigma^a)$ at an arbitrary point, we get a one-form co-vector $(\Sigma(\sigma^a), d\theta)$ in the direction of the normal to this surface at that point. Then we set $(\Sigma(\sigma^a), d\theta) = 0$, we obtain an equation of continuity on $\Sigma_t(\sigma^a)$.

Now taking the exterior derivative of Equation (15), we obtain

$$(16\pi G)^{1/2} d\theta = d\left(E_i \wedge DA^i\right) = (DE_i) \wedge DA^i - E_i \wedge DDA^i, \tag{16}$$

where $E_{0i} = 0$. The tri-tangent basic on $\Sigma_t(\sigma^a)$ is $\partial_a \wedge \partial_b \wedge \partial_c$, we rewrite it as $(1/3!)\varepsilon^{abc}\partial_a\partial_b\partial_c$. The projection of $d\theta$ onto this basic is

$$(16\pi G)^{1/2}\left(d\theta, \varepsilon^{abc}\partial_a\partial_b\partial_c\right) = \left((DE_i) \wedge DA^i, \varepsilon^{abc}\partial_a\partial_b\partial_c\right) - \left(E_i \wedge DDA^i, \varepsilon^{abc}\partial_a\partial_b\partial_c\right)$$

where $(\cdot,\cdot)$ is contraction pairing defined by

$$\left(V_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \varepsilon^{abc}\partial_a\partial_b\partial_c\right) = \varepsilon^{abc} V_{\mu\nu\rho\sigma} dx^{[\mu}\delta^\nu_c \delta^\rho_b \delta^{\sigma]}_a, \tag{17}$$

where the bracket [....] is anti-symmetrization of the indices. Although DDA^i is zero in $4D$ manifold M, but the contraction pairing of $E_i \wedge DDA^i$ with $3D$ basis $\varepsilon^{abc}\partial_a\partial_b\partial_c$ is not zero as we will see, since we do not sum over the time index $\mu = 0$ as we sum over the spatial indices $\mu = 1,2,3$, because we regard dx^0 as normal to the surface $\Sigma_t(\sigma^a)$.

For cotangent basis $\{dx^\mu\}$ and tangent basis $\{\partial_a\}$, this pairing can be defined simply by using inner product like [14]

$$(dx^\mu, \partial_a) = \delta^\mu_a,$$

in which we consider $dx^a = d\sigma^a$ for $a = 1,2,3$, so $E_{0i} = 0$ regarding to our gauge.

Starting with the first term

$$\left((DE_i) \wedge DA^i, \varepsilon^{abc}\partial_a\partial_b\partial_c\right) = \varepsilon^{abc} D_\mu E_{\nu i} D_\rho A^i_\sigma \left(dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \partial_a\partial_b\partial_c\right)$$

we get

$$\begin{aligned}
&4\left((DE_i) \wedge DA^i, \varepsilon^{abc}\partial_a\partial_b\partial_c\right) = \\
&-\varepsilon^{abc} D_a E_{bi} D_c A^i_\mu dx^\mu + \varepsilon^{abc} D_a E_{bi} D_\mu A^i_c dx^\mu - \varepsilon^{abc} D_a E_{\mu i} D_b A^i_c dx^\mu + \varepsilon^{abc} D_\mu E_{ai} D_b A^i_c dx^\mu.
\end{aligned}$$

The second term is

$$\left(E_i \wedge DDA^i, \varepsilon^{abc}\partial_a\partial_b\partial_c\right) = \varepsilon^{abc} E_{\mu i} D_\nu D_\rho A^i_\sigma \left(dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \partial_a\partial_b\partial_c\right).$$

Doing the same thing, we get

$$4\left(E_i \wedge DDA^i, \varepsilon^{abc}\partial_a\partial_b\partial_c\right) =$$
$$-\varepsilon^{abc}E_{ai}D_bD_cA^i_\mu dx^\mu + \varepsilon^{abc}E_{ai}D_bD_\mu A^i_c dx^\mu - \varepsilon^{abc}E_{ai}D_\mu D_b A^i_c dx^\mu + \varepsilon^{abc}E_{\mu i}D_aD_bA^i_c dx^\mu.$$

Adding the two terms, we obtain

$$4(16\pi G)^{1/2}\left(d\theta, \varepsilon^{abc}\partial_a\partial_b\partial_c\right) =$$
$$-\varepsilon^{abc}D_aE_{bi}D_cA^i_\mu dx^\mu + \varepsilon^{abc}D_aE_{bi}D_\mu A^i_c dx^\mu - \varepsilon^{abc}D_aE_{\mu i}D_bA^i_c dx^\mu + \varepsilon^{abc}D_\mu E_{ai}D_bA^i_c dx^\mu$$
$$+\varepsilon^{abc}E_{ai}D_bD_cA^i_\mu dx^\mu - \varepsilon^{abc}E_{ai}D_bD_\mu A^i_c dx^\mu + \varepsilon^{abc}E_{ai}D_\mu D_bA^i_c dx^\mu - \varepsilon^{abc}E_{\mu i}D_aD_bA^i_c dx^\mu.$$

We define the curvature by using the covariant derivative from Equation (13) as

$$F^i_{\mu c} = \frac{1}{2}\left(D_\mu A^i_c - D_c A^i_\mu\right) = \frac{1}{2}\left(\partial_\mu A^i_c - \partial_c A^i_\mu + g\varepsilon^i{}_{jk}A^j_\mu A^k_c\right), \tag{18}$$

therefore

$$F^{\rho\nu i} = g^{\rho\mu}g^{\nu c}F^i_{\mu c} = \frac{1}{2}\left(D^\rho A^{\nu i} - D^\nu A^{\rho i}\right);\ D^\rho g^{\mu\nu} = 0.$$

Its Hodge dual on the surface Σ with respect to the coordinates (σ^a) is

$$F^{ai} = \varepsilon^{abc}F^i_{bc}.$$

Also we define the complex two-form field from Equation (3) as

$$\Sigma^{bc}_i = E^{bc}_i = \varepsilon^{bca}E_{ai} = \varepsilon^{abc}E_{ai}.$$

Using them in the last formula, we get

$$4(16\pi G)^{1/2}\left(d\theta, \varepsilon^{abc}\partial_a\partial_b\partial_c\right) = 2\varepsilon^{abc}(D_aE_{bi})F^i_{\mu c}dx^\mu - 2(D_aE_{\mu i})F^{ai}dx^\mu$$
$$+ 2(D_\mu E_{ai})F^{ai}dx^\mu + 2E^{bc}_i D_bF^i_{c\mu}dx^\mu + 2E_{ai}D_\mu F^{ai}dx^\mu - 2E_{\mu i}D_aF^{ai}dx^\mu.$$

And using

$$\varepsilon^{abc}(D_aE_{bi})F^i_{\mu c}dx^\mu = -\varepsilon^{acb}(D_aE_{bi})F^i_{\mu c}dx^\mu = -(D_aE^{ac}_i)F^i_{\mu c}dx^\mu,$$

we obtain

$$2(16\pi G)^{1/2}\left(d\theta, \varepsilon^{abc}\partial_a\partial_b\partial_c\right) =$$
$$-(D_aE^{ac}_i)F^i_{\mu c}dx^\mu - (D_aE_{\mu i})F^{ai}dx^\mu + \qquad (D_\mu E_{ai})F^{ai}dx^\mu - E^{bc}_i D_bF^i_{\mu c}dx^\mu$$
$$+E_{ai}D_\mu F^{ai}dx^\mu - E_{\mu i}D_aF^{ai}dx^\mu.$$

With

$$-(D_aE^{ac}_i)F^i_{\mu c}dx^\mu - E^{bc}_i D_bF^i_{\mu c}dx^\mu = -D_a\left(E^{ac}_i F^i_{\mu c}dx^\mu\right),$$

it becomes

$$2(16\pi G)^{1/2}\left(d\theta, \varepsilon^{abc}\partial_a\partial_b\partial_c\right) = -D_a(E^{ac}_i F^i_{\mu c})dx^\mu - D_a(E_{\mu i}F^{ai})dx^\mu + D_\mu(E_{ai}F^{ai})dx^\mu.$$

As we suggested before, we let the normal of the surface $\Sigma_t(\sigma^a)$ be in direction of the time dx^0, so $\left(d\theta, \varepsilon^{abc}\partial_a\partial_b\partial_c\right)$ is in direction of the time. Therefore we set $\mu = 0$, thus we get

$$2(16\pi G)^{1/2}\left(d\theta, \varepsilon^{abc}\partial_a\partial_b\partial_c\right) = -D_a(E_i^{ac}F_{0c}^i)dx^0 - 2D_a(E_{0i}F^{ai})dx^0 + D_0(E_{ai}F^{ai})dx^0.$$

The vector $\left(d\theta, \varepsilon^{abc}\partial_a\partial_b\partial_c\right)$ is one-form in the direction of the normal to the surface $\Sigma_t(\sigma^a)$. It is zero as we mentioned before, thus we get

$$-D_a(E_i^{ac}F_{0c}^i)dx^0 - D_a(E_{0i}F^{ai})dx^0 + D_0(E_{ai}F^{ai})dx^0 = 0,$$

or

$$-D_a(E_i^{ac}F_{0c}^i) - D_a(E_{0i}F^{ai}) + D_0(E_{ai}F^{ai}) = 0.$$

We write it as

$$D_a(E_i^{ac}F_{c0}^i) - D_a(E_{0i}F^{ai}) + D_0(E_{ai}F^{ai}) = 0.$$

The term $(E_{ai}F^{ai})$ is scalar, so $D_0(E_{ai}F^{ai}) = \partial_0(E_{ai}F^{ai})$, the vector $E_i^{ab}F_{b0}^i$ is a usual vector field on Σ, it does not carry a Lorentz index, so $D_a(E_i^{ab}F_{b0}^i) = \partial_a(E_i^{ab}F_{b0}^i)$ and $E_{0i} = 0$, thus we get

$$\partial_a(E_i^{ab}F_{b0}^i) + \partial_0(E_{ai}F^{ai}) = 0,$$

using $E_{ai}F^{ai} = \frac{1}{2}E_{abi}F^{abi} = \frac{1}{2}\Sigma_{abi}F^{abi}$

$$\partial_a\left(\Sigma_i^{ab}F_{b0}^i\right) + \partial_0\left(\frac{1}{2}\Sigma_{abi}F^{abi}\right) = 0. \tag{19}$$

This equation shows that there is a relation between Σ^i and F^i in the space (Σ^i, F^i) on $3+1$ manifold $R \times \Sigma$. Usually this relation is written as $F^i = \psi^i{}_j\Sigma^j + \bar{\psi}^i{}_j\bar{\Sigma}^j$. We can find that relation by regarding this equation as an equation of continuity with respect to a Lagrangian like $L(F^{0ai}, DA^i)$ that satisfies the action principle $\delta S(DA^i) = 0$ and the invariance under continuous symmetries of GR. Therefore we regard $\frac{1}{2}c\Sigma_{abi}F^{abi}$ as energy density T^{00}, and $c\Sigma_i^{ab}F_{b0}^i$ as momentum density T^{0a}, where c is constant for satisfying the units. Then we search for a suitable Lagrangian and Hamiltonian with canonical relations that correspond to the same continuity equation according to the quantum fields theory, we do this at the flat-space–time limit and generalize it to an arbitrary curved space–time.

In scalar field ϕ theory, the Lagrangian is [15]

$$L(\phi, \partial\phi) = \pi\partial_0\phi - H(\pi, \phi),$$

the conjugate momentum is $\pi = \partial_0\phi = -\partial^0\phi$. The conserved momentum–energy tensor is

$$T^{00} = H(\pi, \phi) = \pi\partial_0\phi - L(\phi, \partial_\mu\phi) \text{ and } T^{0a} = P^a = \partial^0\phi\partial^a\phi = -\pi\partial^a\phi.$$

In the flat limit of the space–time, our momentum–energy tensor is

$$T^{0a} = c\Sigma^a{}_{bi}F^{0bi} = -c\Sigma_i^{ab}F_{0b}^i = c\Sigma_i^{ab}F_{b0}^i,$$

and

$$T^{00} = \frac{1}{2}c\Sigma_i^{ab}F_{ab}^i = \frac{1}{4}ce_i^a e_j^b F_{ab}^{ij}, \tag{20}$$

comparing it with the momentum $\partial^0\phi\partial^a\phi = -\pi\partial^a\phi$, we conclude that our conjugate momentum is $\pi^{bi} \sim -F^{0bi}$ [16–19].

By considering a Lagrangian of the form $L(F^{0ai}, DA^i)$ with corresponding Hamiltonian like $H(\pi_{ai}, DA^i)$ and using the action principle $\delta S(DA^i) = 0$ and the diffeomorphism invariance on the surface $\Sigma_t(\sigma^a)$, we obtain the momentum–energy tensor like [16–18]

$$T^{00} = \pi_{ai}F^{0ai} - L(F^{0ai}, DA^i) \text{ and } T^{0a} = c\pi_{bi}F^{abi}.$$

Comparing them with our momentum–energy tensor $T^{00} = \frac{1}{2}c\Sigma_{abi}F^{abi}$ and $T^{0a} = c\Sigma^a{}_{bi}F^{0bi}$, we conclude

$$T^{0a} = \pi_{bi}F^{abi} = c\Sigma^a{}_{bi}F^{0bi}. \tag{21}$$

In general, the curvature $F^i_{\mu\nu}$ can be written as [20,21]

$$F^i_{\mu\nu} = \psi^i{}_j\Sigma^j_{\mu\nu} + \psi'^i{}_j\bar{\Sigma}^j_{\mu\nu}, \tag{22}$$

so $\Sigma^{\mu\nu}_i F^i_{\mu\nu} = \psi^i{}_i$ since $\Sigma^{\mu\nu}_i\Sigma^j_{\mu\nu} = \delta^j_i$ and $\Sigma^{\mu\nu}_i\bar{\Sigma}^j_{\mu\nu} = 0$.

Using Equation (22) in Equation (21), we get

$$T^{0a} = \pi_{bi}\psi^i{}_j\Sigma^{abj} + \pi_{bi}\psi'^i{}_j\bar{\Sigma}^{abj} = c\Sigma^a{}_{bi}F^{0bi},$$

therefore we set $\psi'^i{}_j = 0$, so $F^i_{\mu\nu} = \psi^i{}_j\Sigma^j_{\mu\nu}$, and $\Sigma^i_{\mu\nu} = (\psi^{-1})^i{}_jF^j_{\mu\nu}$, we obtain

$$\pi_{bi}\psi^i{}_j\Sigma^{abj} = -c\Sigma^{abi}F_{0bi} \text{ so } \pi^{bi} = c(\psi^{-1})^i{}_jF^{0bj} = c\Sigma^{0bi}, \tag{23}$$

which means that Σ^{0bi} is conjugate momentum of A^i_a. Using it in the Hamiltonian $H = T^{00}$:

$$H = \pi_{ai}F^{0ai} - L(F^{0ai}, DA^i) = -c\Sigma_{0ai}F^{0ai} - L(F^{0ai}, DA^i),$$

then using our energy density $\frac{1}{2}c\Sigma_{abi}F^{abi}$, we get the Lagrangian

$$L(E, F^{0ai}, DA^i) = \pi_{ai}F^{0ai} - H = -c\Sigma_{0ai}F^{0ai} - \frac{1}{2}c\Sigma_{abi}F^{abi}. \tag{24}$$

Therefore

$$L(E, F) = -\frac{1}{2}c(\Sigma_{0ai}F^{0ai} + \Sigma_{0ai}F^{a0i}) - \frac{1}{2}c\Sigma_{abi}F^{abi},$$

so

$$L(E, A) = -\frac{1}{2}c\Sigma_{\mu\nu i}F^{\mu\nu i}, \text{ or } L(E, A) = -\frac{1}{2}c\Sigma_{\mu\nu i}F^{\mu\nu i}\sqrt{-g}d^4x,$$

where $\Sigma^i = E^i$ and F^i are defined in Equations (3) and (18). This Lagrangian has a symmetry of the complex group $SO(3)$ and the self-dual of Lorentz group $SO(3,1)$. The contraction is defined by using the metric $g_{\mu\nu} = \eta_{IJ}e^I_\mu e^J_\nu$. This Lagrangian corresponds to self-dual part of Equation (12). To get the total Lagrangian, we add the Hermitian conjugate, we obtain $L_{dual\,GR}(E, A, \bar{E}, \bar{A}) = L_{Left\,GR}(E, A) + L_{Right\,GR}(\bar{E}, \bar{A})$:

$$L_{dual\,GR}(\Sigma, A, \bar{\Sigma}, \bar{A}) = \frac{-c}{2}\Sigma^{\mu\nu}_i F^i_{\mu\nu}e + \frac{-c}{2}\bar{\Sigma}^{\mu\nu}_i\bar{F}^i_{\mu\nu}e, \tag{25}$$

the curvature $\bar{F}^i$ is the Hermitian conjugate $F^i = P^i_{IJ}R^{IJ}$ (10). The Lorentz group is $SO(3,1)$, its Lie algebra is reducible and can be decomposed into two copies of the Lie algebra of $SU(2)$:

$SO(3,1,C) \cong SO(3,C)_{Left} \times SO(3,C)_{Right}$.

The complex connection $A^i = P^i_{IJ}\omega^{IJ}$ takes values in Lie algebra of $SO(3,C)_{Left}$, while its Hermitian conjugate $\bar{A}^i = \bar{P}^i_{IJ}R^{IJ}$ takes values in Lie algebra of $SO(3,C)_{Right}$. Thus the Lorentz invariance is satisfied by the uni-variance under the two groups $SL(2,R)_{Left}$ and $SL(2,R)_{Right}$ [11,12].

To determine the constant c, we write this Lagrangian in the form

$$(16\pi G)^{-1} e^\mu_I e^\nu_J (R_{\mu\nu})^{IJ} e,$$

by using properties of the projection P^i_{IJ}

$$P^{IJ}_i P^i_{KL} + \bar{P}^{IJ}_i \bar{P}^i_{KL} = \frac{1}{2}(\delta^I_K \delta^J_L - \delta^I_L \delta^J_K), \text{ and } P^i_{IJ}\bar{P}^{IJ}_k = 0, \tag{26}$$

the Lagrangian (Equation (25)) becomes $(-c/2)e^\mu_I e^\nu_J (R_{\mu\nu})^{IJ} e$, thus $-c/2 = (16\pi G)^{-1}$. Therefore

$$L_{dual\ GR}(A, \Sigma, \bar{\Sigma}, \bar{A}) = \frac{1}{16\pi G}\Sigma^{\mu\nu}_i F^i_{\mu\nu} e + \frac{1}{16\pi G}\bar{\Sigma}^{\mu\nu}_i \bar{F}^i_{\mu\nu} e, \tag{27}$$

this Lagrangian is similar to the Plebanisky Lagrangian, but it is not multiplied by the imaginary number i and does not include the cosmological constant term.

4. Yang–Mills Theory of Gravity

By regarding the local Lorentz symmetry as a gauge symmetry with spin connection ω^{IJ} (or A^i) as gauge fields, we recognize Yang–Mills theory in gravity. But not full gravity, since in the Yang–Mills theory, the variables are connections and conserved currents, while in the gravity the metric is also variable. The local Lorentz symmetry generates locally conserved currents, and those currents are coupled to spin connection ω^{IJ}. This makes the local Lorentz symmetry a gauge symmetry with the Lorentz group as a gauge group. Also, these currents must be conserved and vanish in the vacuum.

From the formula $F^i = \psi^i{}_j \Sigma^j + \bar{\psi}^i{}_j \bar{\Sigma}^j$, we can get the inversion $\Sigma^i = \xi^i{}_j F^j + \bar{\xi}^i{}_j \bar{F}^j$ by inserting it back, we obtain

$$\psi^i{}_j \xi^j{}_k + \bar{\psi}^i{}_j \bar{\xi}^j{}_k = \delta^j_k \text{ and } \psi^i{}_j \bar{\xi}^j{}_k + \bar{\psi}^i{}_j \bar{\xi}^j{}_k = 0.$$

We can get the equation of motion $\Sigma^i = \xi^i{}_j F^j + \bar{\xi}^i{}_j \bar{F}^j$ from this Lagrangian (Equation (27)) by adding terms like

$$16\pi G L_{dual\ GR}(A, \Sigma, \bar{\Sigma}, \bar{A}) = \Sigma^{\mu\nu}_i F^i_{\mu\nu} e - \psi_{ij}\Sigma^{\mu\nu i}\Sigma^j_{\mu\nu} e - \psi'_{ij}\bar{\Sigma}^{\mu\nu i}\Sigma^j_{\mu\nu} e + C.C,$$

therefore the $\delta L/\Sigma^{\mu\nu}_i = 0$ and $\delta L/\delta F^i_{\mu\nu} = 0$ yields $F^i = \psi^i{}_j \Sigma^j + \bar{\psi}^i{}_j \bar{\Sigma}^j$ and $D\Sigma^i = 0$. But by using properties of the self-dual projection from Equation (26), we obtain $\bar{\Sigma}^{\mu\nu i}\Sigma^j_{\mu\nu} = 0$ and $\Sigma^{\mu\nu i}\Sigma^j_{\mu\nu} = \delta^{ij}$, but this does not change the equations of motion.

But as we will see, if there is a Lorentz current, like the spin current of the spinor field, then $\delta L/\delta F^i_{\mu\nu} \neq 0$, therefore to keep $D\Sigma^i = 0$, and to also keep $De^I = 0$, we add a term like $F_{\mu\nu i}F^{\mu\nu i}e$. This is done in order to insert back $\Sigma^i \sim F^i$ in $\psi_{ij}\Sigma^{\mu\nu i}\Sigma^j_{\mu\nu}$ into the Lagrangian. Therefore we write

$$L_{dual\ GR}(E, \Sigma, \bar{\Sigma}, \bar{A}) = (16\pi G)^{-1}(\Sigma^{\mu\nu}_i F^i_{\mu\nu} e - \psi_{ij}\Sigma^{\mu\nu i}\Sigma^j_{\mu\nu})e + kF^{\mu\nu}_i F^i_{\mu\nu} e + ... + C.C,$$

where k is a constant can relate to a coupling constant of Lorentz current with the spin connection A^i.

This Lagrangian includes $F^{\mu\nu i}F^{i}_{\mu\nu}e$, F^i is given in Equation (18), similarly to Lagrangian in Yang–Mills theory with gauge group $SO(3)$, and the self-dual of Lorentz group $SO(3,1)$. It depends only on the connection A^i, thus it describes the changing of the local Lorentz basis. Therefore $F^{\mu\nu i}F^{i}_{\mu\nu}e$ reads the invariance only under the local Lorentz transformations. Also, it is a topological invariance, so it allows the free propagation of spin connection A^i. But by relating A^i with the triad e^I, and relating e^I with the metric $g_{\mu\nu}$ by $g_{\mu\nu} = \eta_{IJ}e^I_\mu e^J_\nu$, this breaks the free propagation of spin connection A^i as free waves, except in the background approximation of the metric, the result is gravitational waves. Similarly to the free electromagnetic field.

By using the properties of self-dual projection, this Lagrangian can be written using the Riemannian tensor $R(\omega)$ as

$$L = \frac{1}{16\pi G}Re + kR^2 e + ...$$

We find the role of the term $F^{\mu\nu i}F^{i}_{\mu\nu}e$ by including the interaction of mass-less spinor particles with gravity. Since they are massless, their energies are small so that their interaction with the gravitational field e^I is weak, but their interaction with spin connection takes place. The interaction term is

$$\omega^{IJ}_\mu e^\mu_K \bar{\psi}\gamma^K S_{IJ}\psi e = \omega^{IJ}_\mu J^\mu_{IJ} e, \tag{28}$$

where $J^\mu_{IJ} = e^\mu_K J^K_{IJ} = e^\mu_K \bar{\psi}\gamma^K S_{IJ}\psi$ is Lorentz current. If we add this term to the Lagrangian

$$(16\pi G)^{-1} e^\mu_I e^\nu_J (R_{\mu\nu})^{IJ} e,$$

we get

$$(16\pi G)^{-1} e^\mu_I e^\nu_J (R_{\mu\nu})^{IJ} e + \omega^{IJ}_\mu J^\mu_{IJ} e.$$

So the equation of motion for ω^{IJ}_μ is

$$-(16\pi G)^{-1} D_\mu (e^\mu_{[I} e^\nu_{J]} e) + J^\mu_{IJ} = 0.$$

In self-dual formalism, this equation becomes

$$-(16\pi G)^{-1} D_\mu \Sigma^{\mu\nu i} + J^{\nu i} = 0.$$

These two equations say that the Lorentz current is the source for the gravitational field e^I, but this is not right since the energy is the source for it, also we choose $De^I = 0$. Thus, these equations do not hold. But if we use the Lagrangian

$$L = (16\pi G)^{-1} \Sigma^{\mu\nu}_i F^i_{\mu\nu} e + kF^{\mu\nu}_i F^i_{\mu\nu} e + A^i_\mu J^\mu_i e + C.C, \tag{29}$$

the equation of motion for A^i becomes

$$-(16\pi G)^{-1} D_\mu (\Sigma^{\mu\nu i} e) - \frac{1}{2} k D_\mu F^{\mu\nu i} e + J^{\nu i} = 0.$$

Thus we choose $D_\mu(\Sigma^{\mu\nu i} e) = 0$ and

$$-\frac{1}{2} k D_\mu F^{\mu\nu i} + J^{\nu i} = 0. \tag{30}$$

The term $D_\mu F^{\mu\nu i}$ includes only the spin connection A^i, so the Lorentz current $e^\nu_I\bar{\psi}\gamma^I S^i\psi$ contributes as a source for the spin connection A^i, and so interacts with it. This relate to the fact that we can regard the local Lorentz symmetry as a gauge group with spin connection ω^{IJ} (or A^i) as gauge fields. The Lorentz current is conserved since

$$D_\nu D_\mu F^{\mu\nu i} = \frac{1}{2}[D_\nu, D_\mu]F^{\mu\nu i} = -\frac{1}{2}\varepsilon^i{}_{jk}F^j_{\mu\nu}F^{\mu\nu k} = 0 \to D_\nu J^{\nu i} = 0.$$

Furthermore, Equation (30) allows us to calculate a Lorentz current for a given curvature $F^i_{\mu\nu}$, although the curvatures $R^{IJ}_{\mu\nu}$ and $F^i_{\mu\nu}$ are calculated using only the energy–momentum tensor. Although there are Lorentz currents associated with matter, those currents relate to the local Lorentz symmetry.

We need to prove $\nabla^\mu R_{\mu\nu\rho\sigma} = 0$ in the vacuum, where $R_{\mu\nu\rho\sigma}$ is Riemann curvature tensor, it satisfies $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$, $R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$ and $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$. In the vacuum, we have the equality

$$R_{\mu\nu} = constant \times Rg_{\mu\nu},$$

where $R_{\mu\nu}$ is a Ricci tensor and $R = g^{\mu\nu}R_{\mu\nu}$. This equality is equivalent to another equality [11]:

$$R_{\mu\nu} = constant \times Rg_{\mu\nu} \Leftrightarrow {}^*R_{\mu\nu\rho\sigma} = R^*{}_{\mu\nu\rho\sigma},$$

with Hodge operator

$${}^*R_{\mu\nu\rho\sigma} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\mu'\nu'}R_{\mu'\nu'\rho\sigma},\ R^*{}_{\mu\nu\rho\sigma} = \frac{1}{2}R_{\mu\nu\rho'\sigma'}\epsilon^{\rho'\sigma'}{}_{\rho\sigma},$$

the anti-symmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ is the volume four-form for metric $g_{\mu\nu}$. Therefore in the vacuum, we have

$$\epsilon_{\mu\nu}{}^{\mu'\nu'}R_{\mu'\nu'\rho\sigma} = R_{\mu\nu\rho'\sigma'}\epsilon^{\rho'\sigma'}{}_{\rho\sigma},\ \text{so}\ \epsilon_{\mu\nu\mu'\nu'}R^{\mu'\nu'}{}_{\rho\sigma} = R_{\mu\nu}{}^{\rho'\sigma'}\epsilon_{\rho'\sigma'\rho\sigma},$$

acting by ∇_γ on both sides, with $\nabla_\gamma\epsilon^{\mu\nu\rho\sigma} = 0$ we get

$$\epsilon_{\mu\nu\mu'\nu'}\nabla_\gamma R^{\mu'\nu'}{}_{\rho\sigma} = \nabla_\gamma R_{\mu\nu}{}^{\rho'\sigma'}\epsilon_{\rho'\sigma'\rho\sigma}.$$

The summing here is over μ', ν', ρ' and σ', while μ, ν, ρ, σ and γ are fixed. Then multiplying both sides by $\epsilon^{\gamma\mu\nu\alpha}$:

$$\epsilon^{\gamma\mu\nu\alpha}\epsilon_{\mu\nu\mu'\nu'}\nabla_\gamma R^{\mu'\nu'}{}_{\rho\sigma} = \epsilon^{\gamma\mu\nu\alpha}\nabla_\gamma R_{\mu\nu}{}^{\rho'\sigma'}\epsilon_{\rho'\sigma'\rho\sigma}.$$

We note that by summing over γ, μ, ν in $\epsilon^{\gamma\mu\nu\alpha}\nabla_\gamma R_{\mu\nu}{}^{\rho'\sigma'}$ for each fixed ρ, σ, α, we obtain the Bianchi identity $\epsilon^{\gamma\mu\nu\alpha}\nabla_\gamma R_{\mu\nu}{}^{\rho'\sigma'} = 0$, therefore

$$\epsilon^{\gamma\mu\nu\alpha}\epsilon_{\mu\nu\mu'\nu'}\nabla_\gamma R^{\mu'\nu'}{}_{\rho\sigma} = 0.$$

It becomes

$$2(\delta^\alpha_{\mu'}\delta^\gamma_{\nu'} - \delta^\gamma_{\mu'}\delta^\alpha_{\nu'})\nabla_\gamma R^{\mu'\nu'}{}_{\rho\sigma} = 0 \to 2\nabla_\gamma R^{\alpha\gamma}{}_{\rho\sigma} - 2\nabla_\gamma R^{\gamma\alpha}{}_{\rho\sigma} = -4\nabla_\gamma R^{\gamma\alpha}{}_{\rho\sigma} = 0,$$

so $\nabla^\mu R_{\mu\nu\rho\sigma} = 0$ in the vacuum. Therefore the Lorentz current $J^{IJ}_\nu = \nabla^\mu R^{IJ}_{\mu\nu} = e^{I\rho}e^{J\sigma}\nabla^\mu R_{\mu\nu\rho\sigma}$ vanish in the vacuum. Using the self-dual projection, we find

$$J^i_\nu = \nabla^\mu F^i_{\mu\nu} = P^i_{IJ}\nabla^\mu R^{IJ}_{\mu\nu}$$

also vanish in the vacuum. Therefore this current associates only with matter. We can also prove this by using the formula $F^i = \psi^i{}_j\Sigma^j + \bar{\psi}^i{}_j\bar{\Sigma}^j$, and by setting $\bar{\psi}^i{}_j = 0$ in the vacuum [11]. Using

$$(*d * F^i)_\mu = \nabla^\nu F^i_{\nu\mu},$$

with $*F^i = \psi^i{}_j(*\Sigma^j) + \bar{\psi}^i{}_j(*\bar{\Sigma}^j)$, and

$$* \Sigma^j = -i\Sigma^j, \;\; * \bar{\Sigma}^j = i\bar{\Sigma}^j, \tag{31}$$

Hodge operator here is with respect to the metric $g_{\mu\nu}$, we get

$$\nabla^\nu F^i_{\nu\mu} = i(*(-d\psi^i{}_j\Sigma^j + d\bar{\psi}^i{}_j\bar{\Sigma}^j))_\mu.$$

The first term becomes

$$d(\psi^i{}_j\Sigma^j) = d(\psi^i{}_j\Sigma^j + \bar{\psi}^i{}_j\bar{\Sigma}^j) - d(\bar{\psi}^i{}_j\bar{\Sigma}^j), \text{ so } d(\psi^i{}_j\Sigma^j) = dF^i - d(\bar{\psi}^i{}_j\bar{\Sigma}^j),$$

and by Bianchi identity $dF^i = 0$, we obtain $d(\psi^i{}_j\Sigma^j) = -d(\bar{\psi}^i{}_j\bar{\Sigma}^j)$. Therefore

$$\nabla^\nu F^i_{\nu\mu} = i(*(d(\bar{\psi}^i{}_j\bar{\Sigma}^j) + d\bar{\psi}^i{}_j\bar{\Sigma}^j))_\mu = 2i(*d(\bar{\psi}^i{}_j\bar{\Sigma}^j))_\mu = -2i(*d(\psi^i{}_j\Sigma^j))_\mu. \tag{32}$$

Since $\bar{\psi}^i{}_j = 0$ in the vacuum, we get $\nabla^\nu F^i_{\nu\mu} = 0$.

Therefore for a Lorentz J^i_μ current that associates with matter, we get

$$\nabla^\nu F^i_{\nu\mu} = 2i(*d(\bar{\psi}^i{}_j)\bar{\Sigma}^j)_\mu = J^i_\mu, \text{ so } -2id\bar{\psi}^i{}_j \wedge \bar{\Sigma}^j = *J^i,$$

Where we used $d\bar{\Sigma}^j = 0$ and $J^i = J^i_\mu dx^\mu$, with property of Hodge dual twice operation on p-form V in n dimensions: $* * V = (-1)^{pq+t}V$, where $q = n - p$, and t is the number of negative eigenvalues of the metric tensor [14]. In our case we have $n = 4,\; p = 3,\; t = 1$.

Same thing we get for $\psi^i{}_j$, (Equation (32)):

$$\nabla^\nu F^i_{\nu\mu} = -2i(*(d\psi^i{}_j \wedge \Sigma^j))_\mu = J^i_\mu, \text{ so } -2id\psi^i{}_j \wedge \Sigma^j = *J^i.$$

Therefore

$$-2i(D_\mu\psi^i{}_j)\Sigma^j_{\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho = J^{i\sigma}\epsilon_{\mu\nu\rho\sigma}dx^\mu \wedge dx^\nu \wedge dx^\rho/3!,$$

so

$$2i(D_\mu\psi^i{}_j)\Sigma^j_{\nu\rho}\epsilon^{\mu\nu\rho\sigma} = J^{i\sigma}.$$

But

$$\Sigma^j_{\nu\rho}\epsilon^{\nu\rho\mu\sigma}/3! = (*\Sigma^j)^{\mu\sigma} = (-i\Sigma^j)^{\mu\sigma},$$

(we used self-dual properties Equation (31)), we obtain

$$(D_\mu\psi^i{}_j)\Sigma^{j\mu\nu} = \frac{J^{i\nu}}{2 \times 3!}. \tag{33}$$

The quantity $2(D_\mu \psi^i{}_j)\Sigma^{j\mu\nu}$ depends on the triads e^I, and on spin connection ω^{IJ}, while $J^{i\nu}$ relates matter. Therefore by solving this equation for a given Lorentz current $J^{i\sigma}$, we get solutions for e^I and ω^{IJ}. But this formula reads only the contributing of Lorentz current in $\psi^i{}_j$, there is a contributing of cosmological constant and matter in symmetric part of ψ; $tr(\psi) = -\Lambda - 2\pi G T$, where T is trace of energy-momentum tensor $T_{\mu\nu}$ [12,22]. Since Lorentz current generates local Lorentz transformation, we expect that $D_\mu \psi^i{}_j$ in Equation (33) is anti-symmetric, this distinguishes contributing of $J^{i\sigma}$ from those of cosmological constant and matter. So we write

$$2(D_\mu \psi_k)\varepsilon^{ki}{}_j \Sigma^{j\mu\nu} = J^{i\nu}, \tag{34}$$

thus $J^{i\nu}$ contributes in anti-symmetric part of $\psi^i{}_j := \psi_k \varepsilon^{ki}{}_j$, where ψ^i is vector field in local Lorentz frame.

Let us write $D_\mu \psi^i = P^i_{IK} J^{IK}_\mu$, with Lorentz current $J^{IK}_\mu = J^{IK}_L e^L_\mu$ defined in Equation (28). The relationship (linear) between J^{IK}_μ and $J^{i\nu} = J^i_I e^{I\nu}$ can be determined by inserting $D_\mu \psi^i = P^i_{IK} J^{IK}_\mu$ in Equation (34). It is easier to solve

$$D_\mu \psi^i = P^i_{IK} J^{IK}_\mu$$

in region away from matter where $J^{IK}_\mu = 0$, so we can solve it in background approximation, $g \approx \eta$, thus $\psi^i \to (\psi^r, \psi^\theta, \psi^\phi)$ in spherically coordinates. If we assume that the vector field ψ^r depends only on the radius r, we get

$$\frac{1}{r^2}\partial_r (r^2 \psi^r(r)) = 0 \to \psi^r(r) = \frac{a^r}{r^2},$$

where a^r is constant vector. Therefore the contributing of Lorentz current $J^{i\nu}$ in the curvature $F^i_{\mu\nu}$ is

$$F^i_{\mu\nu} := \frac{1}{r^2} a_k \varepsilon^{ki}{}_j \Sigma^j_{\mu\nu},$$

therefore the contributing of $J^{i\nu}$ in F^2 is

$$F^i_{\mu\nu} F^{\mu\nu}_i := \frac{2}{r^4} a^2, \text{ with } \Sigma^{\mu\nu}_i \Sigma^j_{\mu\nu} = \delta^j_i.$$

This formula satisfies $\oint_S \left\| F^i_{\mu\nu} \right\| dS = 4\pi\sqrt{2a^2} = constant$, it is similar to electric field $\oint_S \vec{E} \cdot d\vec{S} = Q/\varepsilon_0$. This is similarity between GR and Yang-Mills theory.

The Lorentz current is not associated only with spinor particles, the Lorentz symmetry for arbitrary field produces a global conserved Lorentz current like ([15], section 22)

$$M^{IJK} = x^J T^{IK} - x^K T^{IJ}.$$

T^{IK} is energy-momentum tensor in flat space–time. Locally we write this as

$$M^{\mu JK} = a^\nu (e^J_\nu T^{\mu K} - e^K_\nu T^{\mu J}),$$

where a^ν is constant, therefore

$$\nabla_\mu M^{\mu JK} = 0;\ \nabla_\mu e^J_\nu = 0,\ \nabla_\mu T^{\mu K} = 0,\ T^{\mu K} = e^K_\nu T^{\mu\nu}.$$

The existence of a conserved spin current J^i_μ that couples to A^i lets us to believe in local Lorentz symmetry as a gauge group with spin connection ω^{IJ} (or A^i) as gauge fields similarly to Yang–Mills theory.

5. Beta Function

We assume that the interaction of mass-less particles with the gravity is dominated by interaction of their spin current (Lorentz current) with the connection A^i_μ. We can use the Lagrangian $F_{\mu\nu i}F^{\mu\nu i}$ to describe the interaction of left-handed fermions with the connection A^i_μ. We choose a representation of $SL(2,C)$ in which we have $K^i = iJ^i$, where K^i are boost generators and J^i are rotation generators [23]. Our connection A^i is a one-form complex given by the self-dual projection $A^i_\mu = P^i_{IJ}\omega^{IJ}_\mu$ of the spin connection ω^{IJ}_μ according to the decomposition $so(3,1:C) = so(3:C) \oplus so(3:C)$ [24]. Therefore

$$A^iJ^i = Re(A^i)J^i + iIm(A^i)J^i = Re(A^i)J^i + Im(A^i)K^i,$$

or $A^iJ^i + A'^iK^i$, where A^i and A'^i are real.

For simplicity let us choose $A'^i = \gamma A^i$, with a constant $\gamma \in R$. Thus the connection becomes

$$A^iJ^i + A'^iK^i = A^iJ^i + i\gamma A^iJ^i = A^i\left(J^i + i\gamma J^i\right) = A^iT^i \in \Omega^1(M, sl(2,C)),$$

with new generators $T^i = J^i + i\gamma J^i = (1+i\gamma)\,J^i \in sl(2,C)$.

Therefore the coupling of the connection A^i with left-handed spinor field is $\frac{1}{2}gA^i_\mu e^\mu_I\psi^+\bar{\sigma}^IT^i\psi e$, where g is coupling constants comes from using the covariant derivative seen in Equation (14). So we write the fermion-gravity Lagrangian as

$$L(A,\psi) = ie^\mu_I\psi^+\bar{\sigma}^I\partial_\mu\psi e + \frac{1}{4}F_{\mu\nu i}F^{\mu\nu i}e + \frac{1}{2}gA^i_\mu e^\mu_I\psi^+\bar{\sigma}^IT^i\psi e.$$

Using the metric $g_{\mu\nu} = \eta_{IJ}e^I_\mu e^J_\nu$, we obtain

$$F_{\mu\nu i}F^{\mu\nu i} = g_{\mu\mu'}g_{\nu\nu'}F^{\mu'\nu' i}F^{\mu\nu i} = e^I_\mu e_{I\mu'}e^J_\nu e_{J\nu'}F^{\mu'\nu' i}F^{\mu\nu i}.$$

In background space–time, we have $e^i_\mu(x) = \delta^i_\mu + h^i_\mu(x)$, so $g_{\mu\nu}(x) = \eta_{IJ}\delta^I_\mu\delta^J_\nu + ...$, where η is the Minkowski metric. Therefore the gravity and spinor Lagrangian approximates to

$$L(A,\psi) = i\psi^+\bar{\sigma}^\mu\partial_\mu\psi + \frac{1}{4}\eta_{\mu'\mu}\eta_{\nu'\nu}F^{\mu'\nu' i}F^{\mu\nu i} + \frac{1}{2}gA^i_\mu\psi^+\bar{\sigma}^\mu T^i\psi + ...$$

The remaining term includes the interaction with the fluctuated gravitational field $h^i_\mu(x)$, this interaction relates with local invariance under diffeomorphism of M. Let us consider only the part

$$L(A,\psi) = i\psi^+\bar{\sigma}^\mu\partial_\mu\psi + \frac{1}{4}\eta_{\mu'\mu}\eta_{\nu'\nu}F^{\mu'\nu' i}F^{\mu\nu i} + \frac{1}{2}gA^i_\mu\psi^+\bar{\sigma}^\mu T^i\psi,$$

which is invariant under local Lorentz transformation, the compatible currents take values in $sl(2,C)$, thus A^i is coupled to Lorentz currents. By that we have included only the invariance under local Lorentz transformation and excluded the local invariance under diffeomorphism of M. Thus we describe the GR using Lorentz frames as vector bundle over basis space M with connection $A^i \in T^*_pM \times sl(2,C)$.

It is similar to the Lagrangian of Yang–Mills theory for spinor field, but with the generators $\frac{1}{2}T^i = \frac{1}{2}(1+i\gamma)\,J^i$, so to get results from the usual theory, we just replace the generators J^i with $\frac{1}{2}(1+i\gamma)\,J^i$.

For example, to get the beta function $\beta(g) = \partial g/\partial ln(M)$ for our Lagrangian, we use the beta function of Yang–Mills theory for spinor field with symmetry group like $SU(N)$, it is given by [15,25]

$$\beta(g) = -\left[\frac{11}{3}T(A) - \frac{4}{3}n_f T(R)\right]\frac{g^3}{16\pi^2} + O(g^5).$$

The numbers $T(A)$ and $T(R)$, are given in

$$Tr(J_R^a J_R^b) = T(R)\delta^{ab} \; and \; Tr(J_A^a J_A^b) = T(A)\delta^{ab},$$

where J_R^a are generators for the fundamental representation of $SU(N)$ and J_A^a are generators for the adjoint representation. To get them for our Lagrangian, we have to start with the commutation relation $[J^a, J^b] = if^{abc}J^c$ and note that we can multiply both sides by $\frac{1}{4}(1+i\gamma)^2$ to get

$$\left[\frac{1}{2}(1+i\gamma)J^a, \frac{1}{2}(1+i\gamma)J^b\right] = i\frac{1}{2}(1+i\gamma)f^{abc}\frac{1}{2}(1+i\gamma)J^c,$$

thus we get new anti-symmetric structure constants $\frac{1}{2}(1+i\gamma)f^{abc}$, although the new generators are not hermitian, but this does not violates the methods of deriving the beta function, the necessary thing in deriving it is keeping $T(A)$, $T(R)$ and f^{abc} constants [15,26,27]. Anyway, we will absorb the modification factor $\frac{1}{2}(1+i\gamma)$ into the coupling constant g, so we have $SU(2)$ gauge group with complex coupling constant like $\frac{1}{2}(1+i\gamma)g$. Therefore we obtain

$$Tr[J_R^a J_R^b] = T(R)\delta^{ab} \rightarrow Tr\left[\frac{1}{2}(1+i\gamma)J_R^a\frac{1}{2}(1+i\gamma)J_R^b\right] = \frac{1}{4}(1+i\gamma)^2 T(R)\delta^{ab}$$

and

$$Tr[J_A^a J_A^b] = T(A)\delta^{ab} \rightarrow Tr\left[\frac{1}{2}(1+i\gamma)J_A^a\frac{1}{2}(1+i\gamma)J_A^b\right] = \frac{1}{4}(1+i\gamma)^2 T(A)\delta^{ab}.$$

Thus to get beta function for our Lagrangian, we replace $T(A)$ with $\frac{1}{4}(1+i\gamma)^2 T(A)$ and $T(R)$ with $\frac{1}{4}(1+i\gamma)^2 T(R)$, so using this in the beta function

$$\beta(g) = -\left[\frac{11}{3}T(A) - \frac{4}{3}n_f T(R)\right]\frac{g^3}{16\pi^2} + O(g^5),$$

we get

$$\beta(g) = -\frac{1}{4}(1+i\gamma)^2\left[\frac{11}{3}T(A) - \frac{4}{3}n_f T(R)\right]\frac{g^3}{16\pi^2} + O(g^5).$$

For the group $SU(2)$, we have $T(A) = 2$ and $T(R) = \frac{1}{2}$, so for $n_f = 1$, we obtain beta function like

$$\beta(g) = -\frac{5}{3}(1+i\gamma)^2\frac{g^3}{16\pi^2} + O(g^5).$$

Taking in consideration the first statement we obtain

$$\frac{\partial g}{\partial ln(M)} = -\frac{5}{3}(1+i\gamma)^2\frac{g^3}{16\pi^2},$$

which can be solved for energy scale M as

$$g^{-2}(M) = \frac{10}{3 \cdot 16\pi^2} (1 + i\gamma)^2 \, ln \left(\frac{M}{M_0}\right) + g^{-2}(M_0).$$

Usually the coupling constant is real, so for our one, we consider that the interaction strength is governed by the real part of the coupling constant g, this is

$$g_{int}^{-2}(M) = \frac{10}{3 \cdot 16\pi^2} \left(1 - \gamma^2\right) ln \left(\frac{M}{M_0}\right) + g_{int}^{-2}(M_0),$$

thus the behavior of interaction of the left-fermions with the spin connection A^i according to our Lagrangian depends on γ, if $\gamma > 1$, then $\beta(g) > 0$, which means that this interaction becomes stronger as the energy increases, until breaking the perturbation at some energy scale. It is natural to consider $\gamma > 1$ since we expect $A'^i = A^i_{boost} > A^i_{rotation} = A^i$ in nature; the gravity effects the particles by changing their energies (like accelerating a particle via a gravitational field) not by changing their angular momentums. So from $A' = \gamma A$, we have $\gamma > 1$ for this case.

But when does the case $\gamma > 1; \beta(g) < 0$ appear? it appears when $A^i_{rotation} > A^i_{boost}$, in this case, the gravity induces a rotation of the inertial frame (the Lorentz frame moves with a particle, or the Lorentz frame in which the particle has a constant speed) more than changing the energy of that particle (accelerating). This occurs when there is the smallest distance between two particles which interact by their gravitational field. At this distance, the velocities of the two particles are constant, so the interaction by the gravity is dominated by changing their angular momentum, thus $A' > A$ so $\gamma > 1$. This situation appears in the back holes; which is a confinement of $\beta(g) < 0$.

6. Conclusions

We have considered the spacial Lorentz orthonormal basis (e^1, e^2, e^3) as an element in a vector bundle with real spin connection ω^{ij}, which takes values in the Lie algebra of group $SO(3)$ or $SU(2)$. We considered this vector bundle as a tangent vector bundle on the $3D$ hypersurface of constant time $\Sigma_t(\sigma^a)$, this allowed us to define three-form $\theta = \text{constant} \times \varepsilon_{ijk} E^i \wedge R^{jk}$ in the phase space (E^a_i, ω^{ij}_a) on this surface. By arbitrary transformation of x^μ, the three-form θ becomes on M, but $(\Sigma(\sigma^a), d\theta) = 0$ is always satisfied. By doing the same thing, we obtained the equation $(\Sigma(\sigma^a), d\theta) = 0$ using self-dual and anti-self-dual formalism. This equation produces an equation of continuity on the hypersurface $\Sigma_t(\sigma^a)$. We found that Σ^{0a}_i is a conjugate momentum of A^i_a where $\Sigma^{ab}_i F^i_{ab}$ is its energy density. We saw that we have to include the term $L(DA^i)ed^4x = (1/4) F_{\mu\nu i} F^{\mu\nu i} ed^4x$ in the GR Lagrangian, since there is a conserved spin current that couples to A^i. This is the similarity between GR and Yang–Mills theory of gauge fields. If we can solve the GR equations using only spin current, we may consider GR as Yang–Mills theory of gauge fields on a curved space–time manifold with spin connection A^i as a gauge field.

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